

Assignment 3: due Sept 10

Problems to be submitted:

8. The solution to a given differential equation is found as the sum (also often referred to as *superposition* in other maths or physics courses) of an infinite series

$$S(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \cos(n\theta), n = 0, 1, 2, \dots$$

where r and θ are polar coordinates, and $r < R$, R being a positive real constant. We can simplify the solution as done in the class. Define a complex $z = (r/R)e^{i\theta}$, find the sum of the geometric series

$$S_z = \sum_{n=0}^{\infty} z^n,$$

and from there find the simplified form of $S(r, \theta)$. [Also practise Section 11, Problem 17, but do not submit the practice problem.]

9. In the class last week, we used a complex solution for the pendulum problem,

$$\phi = \phi_0 e^{i\omega t},$$

where ϕ_0 and $\omega = \sqrt{g/l}$ are real, standing for the amplitude and frequency of the **small-angle** oscillation.

(a) For physical quantities, we take the real part of the complex solution. Present your **physical solution**, and graph it as a function of t .

(b) Take the time derivative of the complex solution of ϕ to find the angular speed of the pendulum. Again, present the **physical solution** of the angular speed from your complex solution, and graph the angular speed of the pendulum as a function of t .

(c) Now take the time derivative of the **physical solution** in (a) to find the angular speed of the pendulum. Is your answer the same as in (b)?

(d) The solution in (a) describes oscillation of the pendulum in vacuum without friction. In reality, we have to take into account the friction against motion. The solution for this problem is also simply obtained using **complex notations**

$$\phi = \phi_0 e^{i\omega t}.$$

In this case, ω is a complex given by $\omega = \omega_d + i\gamma$, both ω_d and γ are real positive constants (in the Mechanics class later on, you will find what physics determines these two constants). Again, present the **physical solution** by using the real part of the solution, and graph it as a function of time t for $\gamma \ll \omega_d$. Describe how the period and amplitude of the oscillation change with time.

This kind of oscillation is called damped oscillation.

10. In class and in Problem 8, we have shown and used the principle that the real (imaginary) of the sum is the the sum of the real (imaginary); note that the integral operation in the practice problem is equivalent to a sum. These operations are **linear**, and the principle applies to **linear** operations. Multiplication or division or power is **not** a linear operation. So it is **not** true to state that the real (imaginary) of the multiplication is the multiplication of the real (imaginary). You will show this in the following two ways.

(a) For two complex variables, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where x_1, x_2, y_1, y_2 are real, show that $Re[z_1 \cdot z_2] \neq Re[z_1] \cdot Re[z_2]$.

(b) Express the two complex variables as $z_1 = r_1 e^{i\theta_1}$, and $z_2 = r_2 e^{i\theta_2}$, again show that $Re[z_1 \cdot z_2] \neq Re[z_1] \cdot Re[z_2]$.

11. In the laboratory, we often need to find the electrostatic potential inside a box. In a rectangular box, physicists found that electrostatic potential has the form

$$V(x) = Ae^{\frac{x}{L}} + Be^{-\frac{x}{L}},$$

x is the distance from the center of the box. They also measured the potential at the two edges $x = \pm L$ of the box: at $x = L$, $V = V_0$; at $x = -L$, $V = -V_0$. V_0 is a real positive constant.

(a) Find the constants A and B.

(b) Re-write the solution as

$$V(x) = C \sinh(x/L) + D \cosh(x/L).$$

Find the constants C and D .

(c) The electrostatic field inside the box is the gradient of the potential,

$$E\hat{x} = -\frac{dV(x)}{dx}\hat{x}.$$

Find the electric field by taking the derivative of hyperbolic functions, i.e., your solution in (b).

(d) In your solution in (b), one of the constants C or D is zero; in retrospect, can we predict it without solving for C and D based on the properties of hyperbolic functions?

(e) Graph $V(x)$ and $E(x)$, respectively, as function of x .

(Also practise Section 12, Problems 5-11 to get familiar with hyperbolic functions and their properties; do not submit practice problems.)

12: Boas Chapter 3, Section 2, Problems 6, 10, 14.

Solution:

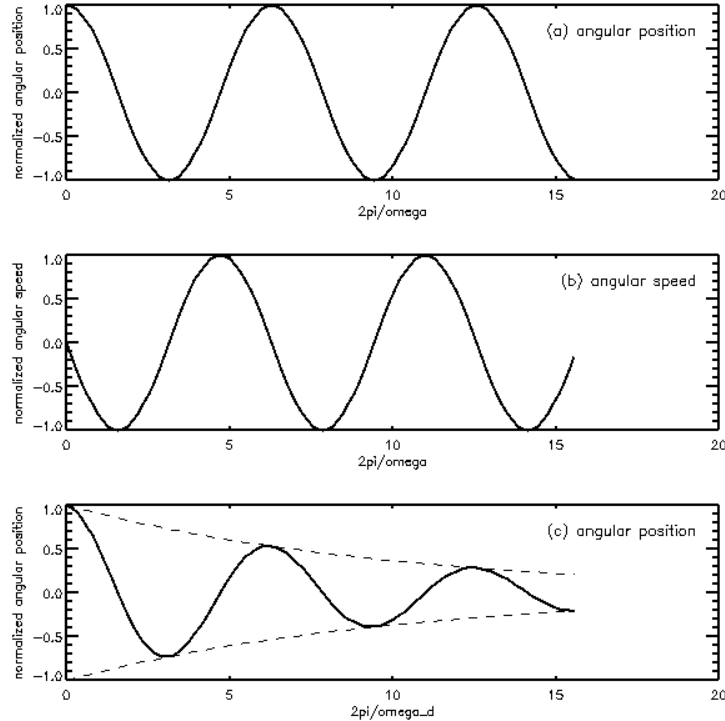
8. [4 points] Define a complex $z = (r/R)e^{i\theta}$, it is seen that the real of the sum of z^n is the solution. For $r < R$, the sum of the geometric series is

$$S_z = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \frac{1}{1 - \frac{r}{R} \cos \theta - i \frac{r}{R} \sin \theta} = \frac{1 - \frac{r}{R} \cos \theta + i \frac{r}{R} \sin \theta}{\left(1 - \frac{r}{R} \cos \theta\right)^2 + \left(\frac{r}{R} \sin \theta\right)^2}.$$

The real part of S_z is therefore,

$$S(r, \theta) = \text{Re}[S_z] = \frac{1 - \frac{r}{R} \cos \theta}{\left(1 - \frac{r}{R} \cos \theta\right)^2 + \left(\frac{r}{R} \sin \theta\right)^2} = \frac{R^2 - rR \cos \theta}{R^2 + r^2 - 2rR \cos \theta}.$$

9. [2+2+1+3 points] (a) The physical solution is the real part of the complex solution $\phi = \phi_0[\cos(\omega t) + i \sin(\omega t)]$, so the physical solution is $\phi_0 \cos(\omega t)$. It is graphed in Figure 1a.
 (b) The time derivative of the complex solution is $i\omega\phi_0 e^{i\omega t} = -\omega\phi_0 \sin(\omega t) + i\omega\phi_0 \cos(\omega t)$. The physical solution is the real part of the solution, $\dot{\phi} = -\omega\phi_0 \sin(\omega t)$. It is graphed in Figure 1b.
 (c) We can derive the angular speed from the solution in (a), $\dot{\phi} = -\omega\phi_0 \sin(\omega t)$, same as in (b).
 (d) In this case, the physical solution is the real of $\phi_0 e^{-\gamma t} [\cos(\omega_d t) + i \sin(\omega_d t)]$, or $\phi = \phi_0 e^{-\gamma t} \cos(\omega_d t)$. It is an oscillation with decaying amplitude, as graphed in Figure 1c. For $\gamma \ll \omega_d$, the period of the oscillation does not vary much with time, and is roughly $2\pi/\omega_d$, and the amplitude of the oscillation decays with time exponentially.



10. [2+2 points] (a) $Re[z_1 \cdot z_2] = Re[(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)] = x_1x_2 + y_1y_2$, and $Re[z_1] \cdot Re[z_2] = x_1x_2$. So the real of the multiplication is not the multiplication of the real.

(b) $Re[z_1 \cdot z_2] = Re[r_1r_2e^{i(\theta_1+\theta_2)}] = r_1r_2 \cos(\theta_1 + \theta_2)$, and $Re[z_1] \cdot Re[z_2] = Re[r_1e^{i\theta_1}] \cdot Re[r_2e^{i\theta_2}] = r_1r_2 \cos(\theta_1) \cos(\theta_2)$. Again, it is shown that the two are not the same.

11. [2+2+2+1+2] (a) We will find A and B using the measurements at the two edges, $V_0 = Ae + Be^{-1}$, and $-V_0 = Ae^{-1} + Be$; solving these two equations, we find

$$A = \frac{V_0}{e - e^{-1}}, \quad B = -\frac{V_0}{e - e^{-1}}.$$

So the solution is

$$V(x) = \frac{V_0 (e^{x/L} - e^{-x/L})}{e - e^{-1}}.$$

(b) Now we can find C and D using hyperbolic functions, $V_0 = C \sinh(1) + D \cosh(-1) = C \sinh(1) + D \cosh(1)$, and $-V_0 = C \sinh(-1) + D \cosh(1) = -C \sinh(1) + D \cosh(1)$, leading to

$$C = \frac{V_0}{\sinh(1)}, \quad D = 0.$$

So the solution is

$$V(x) = \frac{V_0 \sinh\left(\frac{x}{L}\right)}{\sinh(1)}.$$

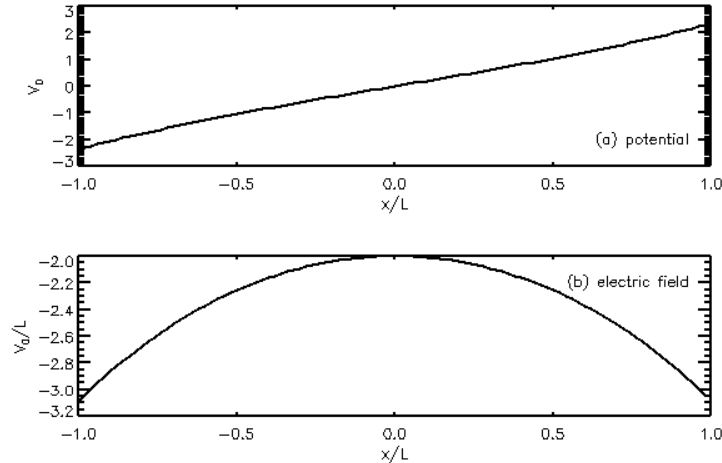
It can be easily shown that this solution is the same as the solution in (a).

(c) Using the solution in (b), we find the electric field

$$E = -\frac{dV(x)}{dx} = -\frac{V_0 \cosh\left(\frac{x}{L}\right)}{L \sinh(1)}.$$

(d) It is easier to find solution in (b); in retrospect, the potential at the two edges $\pm L$ has opposite signs, implying that it is an odd function. The hyperbolic sine function is odd, and hyperbolic cosine function is even, so $D = 0$.

(e) The potential and electric field is plotted in Figure 2, showing that the potential grows from $-L$ to L , and electric field is always negative, but symmetric about the center of the box.



12. [1+2+2] Problem 6: the number of equations is smaller than the number of unknowns, so it is clear that the equations are not solvable. We may further conduct the row reduction by subtracting from the second row two times the first row, leading to:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 2 & -2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

It is clear that $x = 1$, but y and z cannot be solved with unique solutions.

Problem 10: here we will conduct row reduction by (a) subtracting from second row two times the first row, and subtracting from the third row three times the first row, to remove x term in these two rows; (b) subtracting from the third row two times the second row to remove y term in the third row.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & -2 & -1 \\ 3 & 4 & -3 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -2 & 0 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (2)$$

We see that the last row becomes $0x + 0y + 0z = -1$, which is inconsistent. So there is no solution to this set of equations. That's clear even before the final step ..

Problem 14: first we swap the first and second row; then (a) we subtract from the second row two times the first row, and subtract from the third row four times the first row to remove x terms in these two rows; (b) we subtract from the third row the second row.

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 3 & -1 & -2 \\ 4 & 7 & -3 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -1 & 1 & -10 \\ 0 & -1 & 1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -1 & 1 & -10 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad (3)$$

Again, the last row is not consistent, so there is no solution to this set of equations. Again, that's clear before the final step.