

Exam-1 on September 22, in class

Assignment 4: due Sept 17

Problems to be submitted:

13. (a) Find the determinant of the coefficient matrix in Boas Chapter 3, Section 2, Problems 10 & 14. Is your result consistent with your conclusion in HW3-Problem 12?
- (b) Use row reduction method to find the solution (x, y, z) for **Boas Chapter 3, Section 2, Problem 12**.
- (c) Use Laplace development and Cramer's rule to find the solution (x, y, z) for this same problem.
- (d) Find the inverse, M^{-1} , of the coefficient matrix M (Baos Eq. 6.13) for this same problem, and confirm that $M^{-1}M = MM^{-1} = I$, where I is the identity or unit matrix.
- (e) From (d), also confirm that $\det(M^{-1}M) = \det(MM^{-1}) = (\det M) \cdot (\det M^{-1})$.
- (f) Find the solution (x, y, z) by $R = M^{-1}K$.
14. A particle's position in a plane is measured in a cartesian coordinate system XY as such: $\vec{r} = \hat{x} + \sqrt{3}\hat{y}$.
- (a) If the coordinate system rotates about the origin by 30° counterclockwise, find the particle's position $\vec{r} = x'\hat{x}' + y'\hat{y}'$ in this new coordinate system $X'Y'$ using the rotation matrix M .
- (b) Plot the vector $\vec{r}$ in the XY frame, and also plot the X' and Y' axis of the $X'Y'$ frame in the same plot. From this figure, you should be able to check whether your solution in (a) is correct.
- (c) Find the inverse of the rotation matrix M , and show that $M^{-1} = M^T$, namely, the rotation matrix M is orthogonal.
- (d) From $MR = R'$, we find $R = M^{-1}R'$; here R and R' is the matrix notation of $\vec{r}$ in the XY and $X'Y'$ coordinates, respectively. Explain what is the meaning of the transformation $R = M^{-1}R'$. And does your inverse matrix M^{-1} make sense?
15. (a) Baos Chapter 3, Section 6, Problem 10.
- (b) For this problem, we can write $A^{-1}AC = A^{-1}AD$. Since $A^{-1}A = I$, then we should arrive at $C = D$, but clearly $C \neq D$. What is the problem?
16. Baos Chapter 3, Section 7, Problems 23, 28.

Solution: 13. $[1+2+2+4+2+1]$ (a) The determinant of Problem 10 is computed from the first row

$$|M| = (-9 + 8) - 2(-6 + 6) - 1(9 - 8) = 0.$$

The determinant of Problem 14 is computed from the first row

$$|M| = (-9 + 7) - 2(-6 + 4) - 1(14 - 12) = 0.$$

In both cases, the equation set is not solvable; the conclusion is consistent with HW3-Problem 12.

(b) Here we will first swap the first and last row, then conduct row reduction by (a) subtracting from second row the first row, and subtracting from the third row two times the first row, to remove x term in these two rows; (b) subtracting from the third row five times the second row to remove y term in the third row; (b) subtracting from the third row five times the second row to remove the y dependence in the third row;

$$\begin{bmatrix} 1 & 0 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 2 & 5 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 5 & -9 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 6 & 6 \end{bmatrix}.$$

We can then solve the equation to get $z = 1, y = 1, x = -2$.

(c) Following the original set of equations, the coefficient matrix is

$$M = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 5 \end{bmatrix}.$$

Its determinant is

$$|M| = (10 - 1) + 5(2 - 5) = -6.$$

Then we find the solution

$$\begin{aligned} x &= -\frac{1}{6}[3(10 - 1) + 5(2 - 5)] = -2, \\ y &= -\frac{1}{6}[2(5 - 6) - 2(5 - 2) + (3 - 1)] = 1, \\ z &= -\frac{1}{6}[(5 - 2) + 3(2 - 5)] = 1. \end{aligned}$$

(d) The inverse of the matrix can be found as

$$M^{-1} = \frac{C^T}{|M|}.$$

Here the coefficient matrix is

$$M = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 5 \end{bmatrix}.$$

The determinant of this matrix is already found in (c). We then find all the cofactor $c_{11} = 5 - 0 = 5$, $c_{12} = -1(5 - 2) = -3$, $c_{13} = -1$, $c_{21} = -1(25) = -25$, $c_{22} = 10 - 1 = 9$, $c_{23} = -1(0 - 5) = 5$, $c_{31} = 10 - 1 = 9$, $c_{32} = -1(4 - 1) = -3$, $c_{33} = 2 - 5 = -3$. Therefore the inverse of M is given by

$$M^{-1} = -\frac{1}{6} \begin{bmatrix} 5 & -25 & 9 \\ -3 & 9 & -3 \\ -1 & 5 & -3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -5 & 25 & -9 \\ 3 & -9 & 3 \\ 1 & -5 & 3 \end{bmatrix}.$$

We can then find

$$M^{-1}M = \frac{1}{6} \begin{bmatrix} -5 & 25 & -9 \\ 3 & -9 & 3 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 5 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -10 + 25 - 9 & -25 + 25 & -5 + 50 - 45 \\ 6 - 9 + 3 & 15 - 9 & 3 - 18 + 15 \\ 2 - 5 + 3 & 5 - 5 & 1 - 10 + 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$MM^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 5 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} -5 & 25 & -9 \\ 3 & -9 & 3 \\ 1 & -5 & 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -10 + 15 + 1 & 50 - 45 - 5 & -18 + 15 + 3 \\ -5 + 3 + 2 & 25 - 9 - 10 & -9 + 3 + 6 \\ -5 + 5 & 25 - 25 & -9 + 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(e) Since $MM^{-1} = M^{-1}M = I$, $|MM^{-1}| = |M^{-1}M| = 1$. From (c), $|M| = -6$. We then find

$$|M^{-1}| = \frac{1}{6^3} [-5(-27 + 15) - 25(9 - 3) - 9(-15 + 9)] = -\frac{1}{6}.$$

Therefore, $|M||M^{-1}| = 1$.

(f) Now we can find the solution by $R = M^{-1}K$,

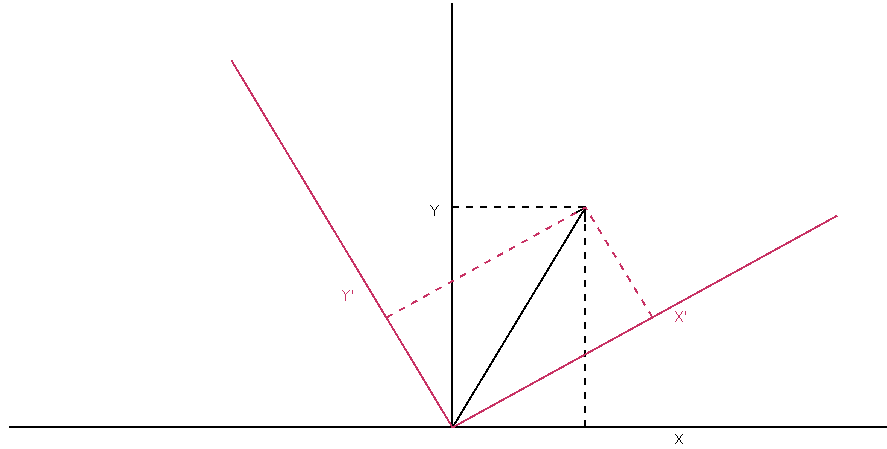
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -5 & 25 & -9 \\ 3 & -9 & 3 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -10 + 25 - 27 \\ 6 - 9 + 9 \\ 2 - 5 + 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

14. [2+2+2+1] (a) We write the vector in matrix form, R and R' , in XY and $X'Y'$ frames, respectively, and then find the transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 30^\circ & \sin 30^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}.$$

So in the new coordinates $X'Y'$, $\vec{r} = \sqrt{3}\hat{x}' + \hat{y}'$.

(b) The figure below shows the vector in XY coordinates (black) and in $X'Y'$ coordinates (red), consistent with (a).



(c) The inverse of the rotation matrix is found by

$$M^{-1} = \frac{C^T}{|M|}.$$

Here $|M| = \cos^2(30^\circ) + \sin^2(30^\circ) = 1$, so we find

$$M^{-1} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}.$$

It is evident that $M^{-1} = M^T$, or the rotation transformation is orthogonal.

(d) The meaning of $R = M^{-1}R'$ is the transformation of the vector's coordinates in $X'Y'$ frame to XY frame which is rotated by -30° from the $X'Y'$ frame. The inverse matrix M^{-1} exactly indicates such a transformation.

15. [2+1] (a) We can compute AC and AD as

$$\begin{aligned} AC &= \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7+4 & 6+6 \\ 21+12 & 18+18 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ 33 & 36 \end{bmatrix}, \\ AD &= \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} -3+14 & 2+10 \\ -9+42 & 6+30 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ 33 & 36 \end{bmatrix}. \end{aligned} \tag{1}$$

So $AC = AD$, though $C \neq D$, and $A \neq 0$.

(b) It is found that the determinant of A is $|A| = 6 - 6 = 0$, therefore A^{-1} does not exist, so we cannot write $A^{-1}AC = A^{-1}AD$.

16. [3+3] Problem 23: to prove that, we need to find the inverse of the matrix

$$M^{-1} = \frac{C^T}{|M|}.$$

The determinant of the matrix is

$$|M| = \left(\frac{1}{2}\right)^2 (3 + 1) = 1,$$

and the cofactors are $c_{11} = -\sqrt{3}/2$, $c_{12} = -1(-1)/2 = 1/2$, $c_{21} = -1(1)/2 = -1/2$, $c_{22} = -\sqrt{3}/2$, so the inverse matrix is

$$M^{-1} = \frac{1}{2} \begin{bmatrix} -\sqrt{3} & -1 \\ 1 & -\sqrt{3} \end{bmatrix} = M^T.$$

Therefore M is orthogonal. Also $|M| = 1 > 0$, so the transformation by this matrix is rotation, no reflection. By observation, it is seen that the rotation angle satisfies $\cos \theta = -\sqrt{3}/2$, $\sin \theta = -1/2$, so the rotation angle is $\theta = 210^\circ$.

Problem 28: For rotation about x -axis, we can write the matrix as

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}.$$

It can be shown that with this matrix, the coordinates has changed to

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \cos \theta - z \sin \theta \\ y \sin \theta + z \cos \theta \end{bmatrix},$$

which keeps x unchanged, and y' and z' as rotated by θ about the x axis.

To combine the rotation with a reflection through the yz plane, we need $x \rightarrow -x$, and the matrix for this transformation is simply

$$M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}.$$