

Assignment 5: due October 1

Problems to be submitted:

17. (a) The position of a particle in the 3D space is $\vec{r} = 2\hat{x} + 3\hat{y} - 1\hat{z}$. Rewrite the vector as $\vec{r} = r\hat{r}$, where r is the distance, and $\hat{r}$ is the unit vector whose magnitude is 1.
- (b) The position of a particle in the 3D space is $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. Rewrite the vector as $\vec{r} = r\hat{r}$, where r is the distance, and $\hat{r}$ is the unit vector whose magnitude is 1. Express your solution as function(s) of x, y, z .
- (c) In Problem-14, write the unit vectors $\hat{x}'$ and $\hat{y}'$ indicating the direction of the rotated frame ($X'Y'$ frame) in terms of their coordinates in the XY frame, and prove that the magnitude of $\hat{x}'$ and $\hat{y}'$ is 1. [hint: here you will rotate the two unit vectors $\hat{x}$ and then $\hat{y}$, respectively, by 30° to get $\hat{x}'$ and $\hat{y}'$, respectively.]

18. In the class, we have stated that we can diagonalize the coefficient matrix and solve algebra equations $MR = K$ by the following (slow!) approach: first, define $R \equiv CR'$, and recast the equations to $DR' = C^{-1}K$. Here C is the matrix made by normalized eigen vectors, and D is the diagonal matrix $D = C^{-1}MC$. We then solve for R' , and then find $R = CR'$. Practise this (slow!) method to find (x, y) for the following equation set following the steps.

$$\begin{aligned}5x - 2y &= 1 \\ -2x + 2y &= 4\end{aligned}$$

- (a) Find the eigen values (which are elements of the diagonal matrix D) of the coefficient matrix.
- (b) Find normalized eigen vectors and construct C .
- (c) Solve for (x', y') from $DR' = C^{-1}K$.
- (d) Solve for (x, y) from $R = CR'$.
- (e) Also show that $\det M = \det D = \lambda_1\lambda_2$.

19. Boas Chapter 3, Section 11, Problems 27, 32.

20. A system of masses and springs are set up like in Boas' Figure 12.1, but the spring constant of the middle spring is $2k$ instead of k . Find the eigen frequencies and eigen vectors of the setup, and explain what is the physical picture of each normal mode.

21. Two particles of masses m and M is attached at the two ends of a spring with spring constant k . The particles can move in only one direction (along the direction of the spring). Find the eigen frequencies and eigen vectors of the system, and explain what is the physical picture of each mode.

Solution:

17. [2+2+4] (a) The magnitude of a unit vector is 1, so we should normalize the vector by its magnitude to get the unit vector. In this case, the magnitude of the position vector is $|\vec{r}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$, so

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{1}{\sqrt{14}}(2\hat{x} + 3\hat{y} - \hat{z}), \quad \vec{r} = r\hat{r} = \sqrt{14} \frac{2\hat{x} + 3\hat{y} - \hat{z}}{\sqrt{14}}.$$

(b) In the same way, we can find

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}}, \quad \vec{r} = \sqrt{x^2 + y^2 + z^2} \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}}.$$

(c) Here we can use the rotation matrix to find $\hat{x}'$ and $\hat{y}'$ by rotating $\hat{x}$ and $\hat{y}$.

$$\begin{aligned} \hat{x}' &= \begin{bmatrix} \cos 30^\circ & \sin 30^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}, \\ \hat{y}' &= \begin{bmatrix} \cos 30^\circ & \sin 30^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}. \end{aligned}$$

So $\hat{x}' = \frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}$, $\hat{y}' = -\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}$. It is seen that $|\hat{x}'| = 1$, $|\hat{y}'| = 1$, so both are unit vectors. Of course, the rotation transformation is an orthogonal transformation, so the rotated vectors should keep the same magnitude.

18. [2+3+3+2+1] (a) We find the eigen values by

$$\det \begin{bmatrix} 5 - \lambda & -2 \\ -2 & 2 - \lambda \end{bmatrix} = (5 - \lambda)(2 - \lambda) - 4 = 0.$$

Solving the quadratic equation, we find two eigen values $\lambda_1 = 1$, $\lambda_2 = 6$.

(b) For $\lambda_1 = 1$, we find the eigen vector r_1 by

$$\begin{bmatrix} 5 - 1 & -2 \\ -2 & 2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0,$$

leading to

$$r_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For $\lambda_2 = 6$, we find the eigen vector r_2 by

$$\begin{bmatrix} 5 - 6 & -2 \\ -2 & 2 - 6 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0,$$

leading to

$$r_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Both eigen vectors are normalized. The conversion matrix is therefore

$$C = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

(c) Now we can solve for $R' = (x', y')$ by $DR' = C^{-1}K$. First the inverse of C is found to be

$$C^{-1} = \frac{C^T}{|C|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Then,

$$DR' = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 7 \\ 6 \end{bmatrix}.$$

This can be easily solved to find

$$R' = \frac{1}{\sqrt{5}} \begin{bmatrix} 7 \\ 6 \end{bmatrix}.$$

(d) Finally we find R by

$$R = CR' = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

So $x = 1, y = 3$, which satisfy the algebra equation.

(e) Also we can find the determinant of M as

$$|M| = (10 - 4) = 6 = |D| = \lambda_1 \lambda_2.$$

19. [3+3] Problem 27: To find the eigen values, we solve

$$\det \begin{bmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{bmatrix} = (\lambda - 1)(\lambda - 3) = 0,$$

leading to $\lambda_1 = 1, \lambda_2 = 3$. For $\lambda_1 = 1$, we find the eigen vector by

$$\begin{bmatrix} 2 - 1 & -1 \\ -1 & 2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0.$$

So the normalized eigen vector is

$$r_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 3$,

$$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0.$$

So the normalized eigen vector is

$$r_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The matrix used to diagonalize M is constructed using eigen vectors,

$$C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

And the matrix M is diagonalized by

$$D = C^{-1}MC = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

So along the direction of the new axis r_1 and r_2 , the amount of deformation is 1 and 3, respectively.

Problem 32: To find the eigen values, we solve

$$\det \begin{bmatrix} 6-\lambda & -2 \\ -2 & 3-\lambda \end{bmatrix} = (\lambda-2)(\lambda-7) = 0,$$

leading to $\lambda_1 = 2, \lambda_2 = 7$. For $\lambda_1 = 2$, we find the eigen vector by

$$\begin{bmatrix} 6-2 & -2 \\ -2 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0.$$

So the normalized eigen vector is

$$r_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For $\lambda_2 = 7$,

$$\begin{bmatrix} 6-7 & -2 \\ -2 & 3-7 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0.$$

So the normalized eigen vector is

$$r_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The matrix used to diagonalize M is constructed using eigen vectors,

$$C = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

And the matrix M is diagonalized by

$$D = C^{-1}MC = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}.$$

So along the direction of the new axis r_1 and r_2 , the amount of deformation is 2 and 7, respectively.

20. [4 points] Let the displacements of the two masses from their equilibrium positions be noted by x and y , we can write down the equations of motion for the two masses separately,

$$\begin{aligned} m\ddot{x} &= -kx + 2k(y - x), \\ m\ddot{y} &= -ky - 2k(y - x). \end{aligned}$$

For homogeneous linear differential equations, we use the solution $x = x_0 e^{i\omega t}$, $y = y_0 e^{i\omega t}$, and the equations of motion become coupled algebra equations,

$$\begin{aligned} (-m\omega^2 + 3k)x - 2ky &= 0, \\ -2kx + (-m\omega^2 + 3k)y &= 0. \end{aligned}$$

For x and y to have non-trivial solutions, the determinant of the coefficient matrix has to be zero, or $(m\omega^2 - 3k)^2 - 4k^2 = (m\omega^2 - 5k)(m\omega^2 - k) = 0$, which yields two eigen frequencies, $\omega_1^2 = k/m$ and $\omega_2^2 = 5k/m$. For the first mode, $\omega_1^2 = k/m$, we find the normalized eigen vector by

$$\begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0.$$

So the normalized eigen vector is

$$r_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This is the mode when both masses move in the same direction by the same amount, in which case, the middle spring is not stretched or compressed; then each particle only receives a restoring force

of $-kx$ or $-ky$, so the frequency of the oscillation is $\sqrt{k/m}$, same as by a single spring. For the second mode, $\omega_2^2 = 5k/m$, we find the normalized eigen vector by

$$\begin{bmatrix} -2k & -2k \\ -2k & -2k \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0.$$

So the normalized eigen vector is

$$r_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This is the mode when two masses move in opposite directions by the same amount, in which case, the middle spring is stretched or compressed by twice of x or y ; then each particle receives the total restoring force of $-5kx$ or $-5ky$, so the frequency of the oscillation is $\sqrt{5k/m}$.

21. [4 points] Let the displacements of the two masses from their equilibrium positions be noted by x and y , we can write down the equations of motion for the two masses separately,

$$\begin{aligned} m\ddot{x} &= -k(x - y), \\ M\ddot{y} &= -k(y - x). \end{aligned}$$

For homogeneous linear differential equations, we use the solution $x = x_0 e^{i\omega t}$, $y = y_0 e^{i\omega t}$, and the equations of motion become coupled algebra equations,

$$\begin{aligned} (-m\omega^2 + k)x - ky &= 0, \\ -kx + (-M\omega^2 + k)y &= 0. \end{aligned}$$

For x and y to have non-trivial solutions, the determinant of the coefficient matrix has to be zero, or $(m\omega^2 - k)(M\omega^2 - k) - k^2 = mM\omega^4 - k(M + m)\omega^2 = 0$, which yields two eigen frequencies, $\omega_1^2 = 0$ and $\omega_2^2 = k/\mu$, where $\mu = Mm/(M + m)$ is the reduced mass. For the first mode, $\omega_1^2 = 0$, we find the normalized eigen vector by

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0.$$

So the normalized eigen vector is

$$r_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This is the mode when both masses move in the same direction by the same amount, in which case, the spring is not stretched or compressed; then particles do not receive any force. So this is the

motion of the whole system (or center of mass) at a constant speed; it is not an oscillation. For the second mode, $\omega_2^2 = k/\mu$, we find the normalized eigen vector by

$$\begin{bmatrix} -\frac{m}{M}k & -k \\ -k & -\frac{m}{M}k \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0.$$

So the normalized eigen vector is

$$r_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{1 + (m/M)^2}} \begin{bmatrix} -1 \\ \frac{m}{M} \end{bmatrix}.$$

This is the mode when two masses move in opposite directions but by different amount. Since $mx_2 + My_2 = 0$, it is seen that the center of mass of the system is still, so x_2 and y_2 are oscillation motions relative to the center of mass. The motion of each particle is a linear combination of two motions: the constant velocity motion of the center of the mass, and oscillation relative to the center of mass.