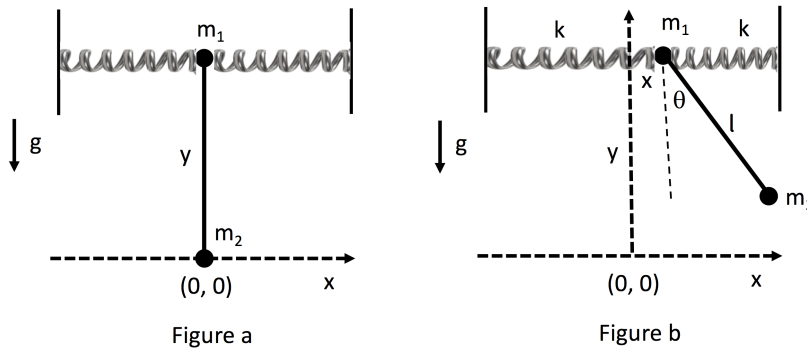


Assignment 6: due October 8

Problems to be submitted:

22. The figure shows a mass particle m_1 between two identical springs with spring constant k , and another mass particle m_2 is attached to m_1 by a massless inextensible string of length l . The gravitational acceleration is a constant g . m_1 can only move along the direction of the springs, and m_2 is a pendulum that can swing back and forth about m_1 . The coordinate system is such that origin is at where m_2 is in equilibrium, as shown in Figure a. When the system is perturbed, the displacement of m_1 is x , and displacement of m_2 is x_2, y_2 in this coordinate system (Figure b). The displacement of m_2 can be also expressed by the angle θ of the pendulum with the vertical, as shown in Figure b.



- Find x_2 and y_2 , respectively, as function of x, θ , which are the two **independent variables** in this coupled system.
- Find the total potential energy of the system $U(x, \theta)$ as a function of x, θ – please consider ALL potential energies in this system.
- Find the two components of the velocity of m_2 , $\dot{x}_2 = \frac{dx_2}{dt}$ and $\dot{y}_2 = \frac{dy_2}{dt}$, respectively, as functions of $x, \theta, \dot{x}, \dot{\theta}$, using chain rules.
- Find the total kinetic energy of the two masses $T = T_1 + T_2$, again as function of $x, \theta, \dot{x}, \dot{\theta}$.
- Can I write the velocity of m_2 as $v_2 = l\dot{\theta}$, and find $T_2 = (1/2)m_2l^2\dot{\theta}^2$? Is the kinetic energy T_2 derived this way same or different from your solution in (d)? If different, which way is the correct way? Explain.
- In your (correct) solution of the potential and kinetic energy functions, there must be $\cos \theta$ or $\sin \theta$ terms. Now we consider small perturbation, i.e., $\theta \ll 1$, expand the $\cos \theta$ or $\sin \theta$ term as power functions of θ about zero, and keep the number of expansion terms so that your potential and kinetic energy functions are only quadratic functions of $x, \theta, \dot{x}, \dot{\theta}$. For example, $x^2, \theta^2, \dot{x}^2, \dot{\theta}^2, x\theta, x\dot{\theta}$ are all quadratic functions.

Note that there should be cross-term in one of your energy functions, otherwise your solution must be wrong!

23. Two particles of masses m_1 and m_2 is attached at the two ends of a spring with spring constant k . The particles can move in only one direction (along the direction of the spring), and their displacements from their equilibrium position are noted as x_1 and x_2 , respectively. Earlier, we have solved the problem with the standard normal modes method. Now we will rewrite the equations of motion with another set of variables defined in the following way:

$$X = \frac{m_1 x_1 + m_2 x_2}{M}, \quad Y = x_2 - x_1.$$

(a) Show that the kinetic energy and the potential energy of this system can be rewritten as

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{Y}^2, \quad U = \frac{1}{2}kY^2,$$

namely, there is no cross-term in the energy functions. Here $M \equiv m_1 + m_2$ is the *total mass*, and $\mu \equiv \frac{m_1 m_2}{M}$ is the *reduced mass*.

(b) Then find the equations of motion with respect to the new variables (X, Y) by

$$M\ddot{X} = -\frac{\partial U}{\partial X}, \quad \mu\ddot{Y} = -\frac{\partial U}{\partial Y}.$$

Show that these two differential equations are no longer coupled.

(c) Explain what is the physical picture of the motion indicated by X and Y – you do not have to solve the differential equations. How is this result related to your solution to HW5-Problem 21?

24. Boas Chapter 4 Section 5, Problem 1.

25. Boas Chapter 4 Section 6, Problem 7.

26. A second-order ordinary differential equation is written as

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dy}{d\theta} \right) + l(l+1)y = 0.,$$

where l is a constant positive integer. Now introduce variable $x = \cos \theta$, rewrite the differential equation with respect to the independent variable x – this is called the Legendre equation.

Solution

22. [2+2+2+2+2+2] (a) $x_2 = x + l \sin \theta$, $y_2 = l(1 - \cos \theta)$.

(b) The total potential energy includes the spring energy and gravitational potential energy,

$$\begin{aligned} U &= \frac{1}{2}kx^2 + \frac{1}{2}kx^2 + m_2gy_2 \\ &= kx^2 + m_2gl(1 - \cos \theta). \end{aligned}$$

(c) $\dot{x}_2 = \dot{x} + l \cos \theta \dot{\theta}$, $\dot{y}_2 = l \sin \theta \dot{\theta}$.

(d) The total kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_2\dot{y}_2^2 \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x} + l \cos \theta \dot{\theta})^2 + \frac{1}{2}m_2(l \sin \theta \dot{\theta})^2 \\ &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\theta}^2 + m_2l \cos \theta \dot{x} \dot{\theta}. \end{aligned}$$

(e) We cannot write the velocity of m_2 as $l\dot{\theta}$, because m_2 also moves together with m_1 . So the kinetic energy $T_2 = \frac{1}{2}m_2l^2\dot{\theta}^2$ is incorrect – **this is the most common, and fatal, mistake in problems of this kind!**

(f) Both energy functions have $\cos \theta$ term. We expand this term as

$$\cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots$$

We will keep up to the second term for the potential energy and up to the first term for the kinetic energy, so that both energy functions are quadratic functions of the independent variables,

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\theta}^2 + m_2l\dot{x}\dot{\theta}, \quad U = kx^2 + \frac{1}{2}m_2gl\theta^2.$$

It is seen that there is no crossterm in the potential energy, but there is a cross-term $\dot{x}\dot{\theta}$ in the kinetic energy, so the motions indicated by the two independent variables (x, θ) are coupled. With the cross-term in the kinetic energy, we cannot write the equations of motion the same way as in the next problem; we will use Lagrange method to find equations of motion, or Euler-Lagrange equations, later in the Mechanics class.

23. [3+2+2] (a) Note that by these definitions, X is the position of the center of mass, and Y is the distance between the two masses. We can find x_1 and x_2 as functions of X, Y as

$$x_1 = X - \frac{m_2}{M}Y, \quad x_2 = X + \frac{m_1}{M}Y.$$

The kinetic energy is therefore

$$\begin{aligned}
 T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \\
 &= \frac{1}{2}m_1\left(\dot{X} - \frac{m_2}{M}\dot{Y}\right)^2 + \frac{1}{2}m_2\left(\dot{X} + \frac{m_1}{M}\dot{Y}\right)^2 \\
 &= \frac{1}{2}(m_1 + m_2)\dot{X}^2 + \frac{1}{2}\frac{m_1m_2}{M}\dot{Y}^2 \\
 &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{Y}^2.
 \end{aligned}$$

And the potential energy is

$$U = \frac{1}{2}k(x_2 - x_1)^2 = \frac{1}{2}kY^2.$$

With the new variables, there is **no cross-term** in potential and kinetic energy functions. So X and Y motions are decoupled.

(b) The two equations of motion now are given by

$$M\ddot{X} = -\frac{\partial U}{\partial X} = 0, \quad \mu\ddot{Y} = -\frac{\partial U}{\partial Y} = -kY.$$

The equations are no longer coupled, i.e., $\ddot{X}$ does not depend on Y , and $\ddot{Y}$ does not depend on X .

(c) The first equation of motion indicates force-free motion of the center of mass, so center of mass will have a motion of constant velocity after the initial perturbation with eigen frequency $\omega_1^2 = 0$. The second equation of motion is an oscillation of the distance between the two particles due to the spring, with the eigen frequency $\omega_2^2 = k/\mu$ – is it the same as your solution in Problem 21?! The motion of each individual mass is a linear combination of these two separate modes of motion, i.e., translational motion of the center of mass and oscillation relative to the center of mass.

24. [2 points] We first take differential of two side of the equation $z = xe^{-y}$ as

$$dz = e^{-y}dx - xe^{-y}dy,$$

so we get

$$\frac{dz}{dt} = e^{-y}\frac{dx}{dt} - xe^{-y}\frac{dy}{dt} = e^{-y}\sinh t + xe^{-y}\sin t.$$

We may also use the method in the class; first take the natural logarithm of the two sides of the equation, $\ln z = \ln x - y$. Then take the differential of both sides, leading to

$$\frac{dz}{z} = \frac{dx}{x} - dy,$$

Therefore, we find

$$\frac{dz}{dt} = \frac{z}{x}\frac{dx}{dt} - z\frac{dy}{dt} = e^{-y}\sinh t + xe^{-y}\sin t,$$

same as above.

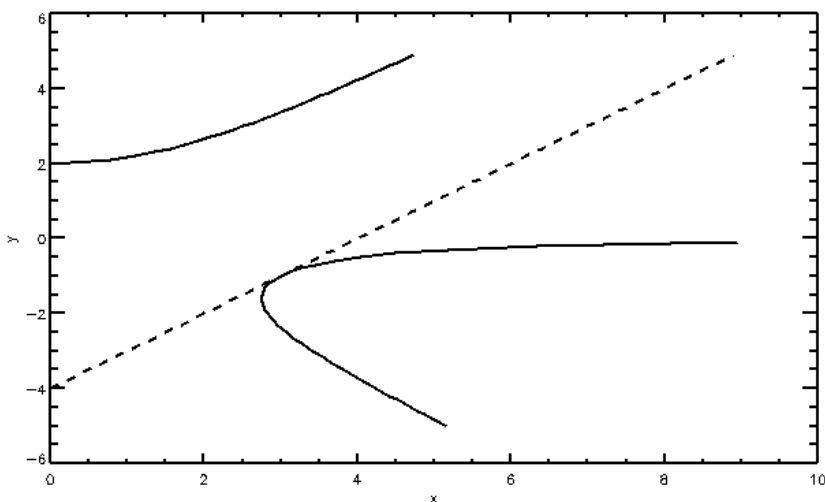
25. [4 points] We take differential of the two sides of the equation

$$3y^2 dy - 2xy dx - x^2 dy = 0,$$

leading to

$$\frac{dy}{dx} = \frac{2xy}{3y^2 - x^2},$$

which is the slope of the curve. At the point $(3, -1)$, the slope is $dy/dx = 1$, and the tangent line crossing the point is $y + 1 = 1(x - 3)$, or $y - x = -4$. The curve and the tangent line are plotted in the figure below.



26. [4 points] Take differential on two sides of the equation $x = \cos \theta$, we find $dx = -\sin \theta d\theta$, so

$$\frac{dx}{d\theta} = -\sin \theta.$$

Then using the chain rule, we find

$$\frac{dy}{d\theta} = \frac{dy}{dx} \frac{dx}{d\theta} = -\sin \theta \frac{dy}{dx}.$$

In the same way,

$$\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{d}{dx} = -\sin \theta \frac{d}{dx}.$$

Take these back to the original differential equation, we find

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dy}{d\theta} \right) = \frac{1}{\sin \theta} (-\sin \theta) \frac{d}{dx} \left[\sin \theta (-\sin \theta) \frac{dy}{dx} \right] = \frac{d}{dx} \left(\sin^2 \theta \frac{dy}{dx} \right) = \frac{d}{dx} \left[(1 - x^2) \frac{dy}{dx} \right].$$

So the original differential equation becomes

$$\frac{d}{dx} \left[(1 - x^2) \frac{dy}{dx} \right] + l(l + 1)y = 0.$$

This is Legendre equation, and its solution is known as Legendre polynomials, which we will use to find electrostatic potential outside a conducting sphere.