Exam-2 on October 22, Wednesday, in class

Assignment 7: due October 17, Friday

Problems to be submitted:

27. In a curvilinear coordinate system, the three unit vectors are **orthogonal**.
(a) In cylindrical coordinate, $\hat{s}\times\hat{\phi}=\hat{z}, \hat{\phi}\times\hat{z}=\hat{s}, \hat{z}\times\hat{s}=\hat{\phi}$. We have shown that $\hat{s}=\cos\phi\hat{x}+\sin\phi\hat{y}$, use the orthogonality to find $\hat{\phi}$.
(b) In spherical coordinates, $\hat{r}\times\hat{\theta}=\hat{\phi}, \hat{\theta}\times\hat{\phi}=\hat{r}, \hat{\phi}\times\hat{r}=\hat{\theta}$. Given $\hat{r}=\sin\theta\cos\phi\hat{x}+\sin\theta\sin\phi\hat{y}+\cos\theta\hat{z}$, and $\hat{\phi}$ same as in (a), find $\hat{\theta}$.
(c) A particle's position is $\vec{r}=r\hat{r}$. Find its velocity $\vec{v}=\frac{d\vec{r}}{dt}$ in spherical coordinates. Hint: recall how we should write the infinitestimal displacement $d\vec{r}$.
(d) Repeat (c), but now find the velocity in cartesian coordinates, and show that your solutions in (c) and (d) are identical.

28. A cone has height H and circular cross-section with its radius $R=2H$ at the bottom. The bottom of the cone is placed in the xy plane, and the axis of the cone is along the z-axis. The mass density of the cone is a constant ρ.
(a) From the definition,

$$\vec{r}_{cm}=\frac{\int\rho\vec{r}dV}{\int\rho dV},$$

we may use cylindrical coordinates to write $\vec{r}=s\hat{s}+z\hat{z}$, and find

$$\vec{r}_{cm}=\frac{\int\rho s dV}{\int\rho dV}\hat{s}+\frac{\int\rho z dV}{\int\rho dV}\hat{z}.$$

Finish the integrals.
(b) We may also find the three components of $\vec{r}_{cm}$ in **cartesian coordinates** by

$$x_{cm}=\frac{\int x\rho dV}{\int\rho dV}, y_{cm}=\frac{\int y\rho dV}{\int\rho dV}, z_{cm}=\frac{\int z\rho dV}{\int\rho dV},$$

and $\vec{r}_{cm}=x_{cm}\hat{x}+y_{cm}\hat{y}+z_{cm}\hat{z}$. Here you will write x, y, z and dV in cylindrical coordinates and finish the integrals.
(c) Compare your solutions in (a) and (b). Are they the same? If not, which solution is correct? Why is the other one incorrect?
(d) Also find the moment of inertial of the cone about z-axis.

29. The electric field by a point charge Q at origin is give by

$$\vec{E}=kQ\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}.$$

Find the work done by this electric field on a test charge q moving from the initial position (x_0, y_0, z_0) to the final position $(2x_0, 2y_0, 2z_0)$, along the following paths.
(a) The path consists of three parts. First, along a straight line with x changing from x_0 to $2x_0$, $y = y_0$, $z = z_0$, i.e. no change in y and z; then along a straight line with y changing from y_0 to $2y_0$, and $x = 2x_0$, $z = z_0$, i.e. no change in x and z, and finally along a straight line with z changing from z_0 to $2z_0$, and $x = 2x_0$, $y = 2y_0$, i.e., no change in x and y.
(b) The path is along a straight line in the direction of the initial positive vector, which is the same direction of the final position vector. Hint: re-write $\vec{E}$, the initial and final position vectors, and the line integral in spherical coordinates.
(c) Compare your solution in (a) and (b): does the result depend on the path of integral?
(d) Evaluate $\vec{\nabla} \times \vec{E}$, and show that $\vec{E}$ is a conservative force.
(e) Show that the electric field $\vec{E}$ is the (minus) gradient of the electric potential, $\vec{E} = -\vec{\nabla}V$, V given by

$$V = \frac{kQ}{r}.$$

(f) Sketch the field lines of $\vec{E}$ and contours of V.

30. Consider a force $\vec{F} = c(y\hat{x} - x\hat{y})$, where c is a positive constant, doing work on a particle as it moves from the initial position (x_0, y_0, z_0) to the final position $(2x_0, 2y_0, 2z_0)$.
(a) Find the work done along the path same as in Problem 29 (a).
(b) Find the work done along the path same as in Problem 29 (b). Here you want to find the relation between y and x along this path, and conduct the integral in cartesian coordinates.
(c) Are the results in (a) and (b) same or different? Find $\vec{\nabla} \times \vec{F}$, and show that this is not a conservative force.
(d) Sketch the field lines of $\vec{F}$.

Solution

27. [1+1+2+4] (a) $\hat{\phi} = \hat{z} \times \hat{s} = \hat{z} \times (\cos\phi\hat{x} + \sin\phi\hat{y}) = -\sin\phi\hat{x} + \cos\phi\hat{y}$.
(b) $\hat{\theta} = \hat{\phi} \times \hat{r} = (-\sin\phi\hat{x} + \cos\phi\hat{y}) \times (\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}) = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$.
(c) The infinitesimal displacement is given by $\vec{dr} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$, so we get

$$\frac{\vec{dr}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} + r\sin\theta\frac{d\phi}{dt}\hat{\phi} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}.$$

(d) In cartesian coordinates, $\vec{dr} = dx\hat{x} + dy\hat{y} + dz\hat{z}$, so

$$\frac{\vec{dr}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}.$$

We will convert the spherical coordinates back to cartesian. First, we write the unit vectors in spherical coordinates into unit vectors in cartesian coordinates, leading to

$$\begin{aligned}
\frac{\vec{dr}}{dt} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi} \\
&= (\sin\theta\cos\phi\dot{r} + r\cos\theta\cos\phi\dot{\theta} - r\sin\theta\sin\phi\dot{\phi})\hat{x} \\
&\quad +(\sin\theta\sin\phi\dot{r} + r\cos\theta\sin\phi\dot{\theta} + r\sin\theta\cos\phi\dot{\phi})\hat{y} \\
&\quad +(\cos\theta\dot{r} - r\sin\theta\dot{\theta})\hat{z}.
\end{aligned}$$

We then use coordinates transformation and chain rules to find $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$ as functions of $\dot{x}$, $\dot{y}$, $\dot{z}$. Here $r = \sqrt{x^2 + y^2 + z^2}$, leading to

$$\dot{r} = \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y} + \frac{z}{r}\dot{z} = \sin\theta\cos\phi\dot{x} + \sin\theta\sin\phi\dot{y} + \cos\theta\dot{z}.$$

From $\cos\theta = z/r$, we find

$$-\sin\theta\dot{\theta} = -\frac{z}{r^2}\dot{r} + \frac{1}{r}\dot{z} \to \dot{\theta} = \frac{\cos\theta}{r\sin\theta}\dot{r} - \frac{1}{r\sin\theta}\dot{z}.$$

Then from $\tan\phi = y/x$, we get

$$\sec^2\phi\dot{\phi} = -\frac{y}{x^2}\dot{x} + \frac{1}{x}\dot{y} \to \dot{\phi} = -\frac{\sin\phi}{r\sin\theta}\dot{x} + \frac{\cos\phi}{r\sin\theta}\dot{y}.$$

Taking these back to $\vec{dr}/dt$ expression, with some algebra and reorganization of terms, it is seen that

$$\begin{aligned}
v_x &= \sin\theta\cos\phi\dot{r} + r\cos\theta\cos\phi\dot{\theta} - r\sin\theta\sin\phi\dot{\phi} \\
&= \left(\sin\theta\cos\phi + \frac{\cos^2\theta\cos\phi}{\sin\theta}\right)\dot{r} - \frac{\cos\theta\cos\phi}{\sin\theta}\dot{z} + \sin^2\phi\dot{x} - \sin\phi\cos\phi\dot{y} \\
&= \frac{\cos\phi}{\sin\theta}\dot{r} - \frac{\cos\theta\cos\phi}{\sin\theta}\dot{z} + \sin^2\phi\dot{x} - \sin\phi\cos\phi\dot{y} \\
&= \frac{\cos\phi}{\sin\theta}(\sin\theta\cos\phi\dot{x} + \sin\theta\sin\phi\dot{y} + \cos\theta\dot{z}) - \frac{\cos\theta\cos\phi}{\sin\theta}\dot{z} + \sin^2\phi\dot{x} - \sin\phi\cos\phi\dot{y} \\
&= \dot{x}.
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
v_y &= \sin\theta\sin\phi\dot{r} + r\cos\theta\sin\phi\dot{\theta} + r\sin\theta\cos\phi\dot{\phi} \\
&= \left(\sin\theta\sin\phi + \frac{\cos^2\theta\sin\phi}{\sin\theta}\right)\dot{r} - \frac{\cos\theta\sin\phi}{\sin\theta}\dot{z} - \sin\phi\cos\phi\dot{x} + \cos^2\phi\dot{y} \\
&= \frac{\sin\phi}{\sin\theta}(\sin\theta\cos\phi\dot{x} + \sin\theta\sin\phi\dot{y} + \cos\theta\dot{z}) - \frac{\cos\theta\sin\phi}{\sin\theta}\dot{z} - \sin\phi\cos\phi\dot{x} + \cos^2\phi\dot{y} \\
&= \dot{y}.
\end{aligned}$$

And then $v_z = \cos\theta\dot{r} - r\sin\theta\dot{\theta} = (\cos\theta - \cos\theta)\dot{r} + \dot{z} = \dot{z}$. Therefore, $\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$.

28. [3+2+1+2] (a) We first find the total mass

$$M = \int \rho dV = 2\pi\rho\int_0^H dz\int_0^{2(H-z)} sds = \frac{4}{3}\pi\rho H^3.$$

Then we find

$$\int \rho s dV = 2\pi\rho\int_0^H dz\int_0^{2(H-z)} s^2 ds = \frac{4}{3}\pi\rho H^4,$$

$$\int \rho z dV = 2\pi\rho\int_0^H zdz\int_0^{2(H-z)} sds = \frac{1}{3}\pi\rho H^4.$$

So

$$\vec{r}_{cm} = H\hat{s} + \frac{1}{4}H\hat{z}.$$

(b) Here we write $x = s\cos\phi, y = s\sin\phi$, so the integrals become

$$x_{cm} = \int \rho x dV = \rho\int_0^H dz\int_0^{2(H-z)} s^2 ds\int_0^{2\pi}\cos\phi d\phi = 0,$$

$$y_{cm} = \int \rho y dV = \rho\int_0^H dz\int_0^{2(H-z)} s^2 ds\int_0^{2\pi}\sin\phi d\phi = 0.$$

The center of mass position is therefore

$$\vec{r}_{cm} = z_{cm}\hat{z} = \frac{1}{4}H\hat{z}.$$

(c) Apparently, from symmetry, we know the solution (b) is correct. Solution (a) is not correct because $\hat{s}$ is not a constant unit vector and its direction changes with the location of dV; therefore $\hat{s}$ cannot be taken out of the integral.

(d) The moment of inertial is given by

$$I = \int r_\perp^2\rho dV = 2\pi\rho\int_0^H dz\int_0^{2(H-z)} s^2 sds = \frac{8}{5}\pi\rho H^5.$$

29. [2+2+1+1+1+2] (a) The force on the charge q is $\vec{F} = q\vec{E}$, so the work done is $W = \int \vec{F}\cdot d\vec{r}$. The line integral consists of three parts

$$
\begin{aligned}
W &= q\int \vec{E}\cdot d\vec{r} = kQq\left[\int_{x_0}^{2x_0} \frac{xdx}{(x^2+y_0^2+z_0^2)^{3/2}} + \int_{y_0}^{2y_0} \frac{ydy}{(4x_0^2+y^2+z_0^2)^{3/2}} + \int_{z_0}^{2z_0} \frac{zdz}{(4x_0^2+4y_0^2+z^2)^{3/2}}\right] \\
&= kQq\left[\frac{1}{(x_0^2+y_0^2+z_0^2)^{1/2}} - \frac{1}{(4x_0^2+y_0^2+z_0^2)^{1/2}} + \frac{1}{(4x_0^2+y_0^2+z_0^2)^{1/2}} - \frac{1}{(4x_0^2+4y_0^2+z_0^2)^{1/2}}\right] \\
&\quad + kQq\left[\frac{1}{(4x_0^2+4y_0^2+z_0^2)^{1/2}} - \frac{1}{(4x_0^2+4y_0^2+4z_0^2)^{1/2}}\right] \\
&= kQq\left[\frac{1}{(x_0^2+y_0^2+z_0^2)^{1/2}} - \frac{1}{(4x_0^2+4y_0^2+4z_0^2)^{1/2}}\right].
\end{aligned}
$$

(b) We write the electric field in spherical coordinates $\vec{E} = kQq\hat{r}/r^2$, and the integral becomes

$$
W = q\int \vec{E}\cdot d\vec{r} = kQq\int \frac{\hat{r}\cdot\hat{r}dr}{r^2} = kQq\int_{r_0}^{r_1} \frac{1}{r^2}dr = kQq\left(\frac{1}{r_0} - \frac{1}{r_1}\right).
$$

(c) The results in (a) and (b) are the same; so the work done by the electric field does not depend on the path.
(d) We can use spherical coordinates to evaluate the curl of $\vec{E}$. Since $\vec{E} = E(r)\hat{r}$, it is easily seen that $\vec{\nabla}\times\vec{E} = 0$.
Therefore, the electric force is a conservative force, and the work done does not depend on the path.
(e) Again, we can find the gradient using the spherical coordinates,

$$
-\vec{\nabla}V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi} = \frac{kQ\hat{r}}{r^2}.
$$

(f) The electric field is radial, and its magnitude decreases with distance by $1/r^2$, as shown in the figure. The contour of V is shown in dashed curves. The electric field is in the direction perpendicular to the V contour.

30. [2+2+1+2] (a) We find the work by

$$
\begin{aligned}
W &= \int \vec{F}\cdot d\vec{r} = \int (F_x dx + F_y dy) = c\int (ydx - xdy) \\
&= c\int_{x_0}^{2x_0} y_0 dx - c\int_{y_0}^{2y_0} 2x_0 dy = -cx_0y_0.
\end{aligned}
$$

(b) Along that path, $y - y_0 = x - x_0$, or $y = x + y_0 - x_0$, so $dy = dx$, then the work done is

$$
\begin{aligned}
W &= \int \vec{F}\cdot d\vec{r} = c\int (ydx - xdy) = c\int_{x_0}^{2x_0} (y-x)dx \\
&= c\int_{x_0}^{2x_0} (y_0 - x_0)dx = cx_0(y_0 - x_0).
\end{aligned}
$$

(c) The results in (a) and (b) are different, so the integra is path dependent. We also find

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} = -2c\hat{z} \neq 0.$$

Therefore, $\vec{F}$ is not a conservative force, and the work done by $\vec{F}$ is dependent on the path.
(d) The field lines of the force are shown in the figure, forming vortices, i.e., curl of the field is not zero.