

## Solutions to Final Exam

1(a) The incident electric field is

$$\vec{E}_I = \frac{E_{I0}}{\sqrt{2}} e^{i(k_x x + k_z z - \omega t)} (\hat{x} - \hat{z})$$

where  $k_x = k_z = k_I/\sqrt{2}$ , and  $k_I = \omega/c$ .

1(b) The incident magnetic field can be determined from Faraday's law  $\vec{B} = \frac{1}{c} \hat{k} \times \vec{E}$ . In this case,  $\hat{k} = 1/\sqrt{2}(\hat{x} + \hat{z})$ , so the magnetic field is

$$\vec{B}_I = \frac{E_{I0}}{c} e^{i(k_x x + k_z z - \omega t)} \hat{y}$$

1(c) From Snell's law,

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_T}{n_I} = \sqrt{2}$$

so  $\sin\theta_T = 1/2$ , and  $\theta_T = 30^\circ$ .

1(d) From the point of view of energy conservation, the total incident light intensity integrated in a surface area is the sum of the reflected and transmitted light intensity integrated in the same area. This is written as  $I_I \cos\theta_I = I_R \cos\theta_R + I_T \cos\theta_T$ . Here,  $I_I = \frac{1}{2} c \epsilon_0 E_{I0}^2$ ,  $I_R = \frac{1}{2} c \epsilon_0 E_{R0}^2$ , and  $I_T = \frac{1}{2} v \epsilon E_{T0}^2 = \frac{1}{2} n_T c \epsilon_0 E_{T0}^2$ . Since  $n_T > 1$ , and  $\cos\theta_T > \cos\theta_I$ , it is seen that there must be  $E_{T0} < E_{I0}$  for energy conservation to hold.

1(e) The boundary conditions require that the tangential component of the electric field is continuous. In the conductor,  $\vec{E}^{\parallel} = \vec{0}$ , so  $\vec{E}_I^{\parallel} + \vec{E}_R^{\parallel} = (E_{Ix} + E_{Rx})\hat{x} = 0$ , yielding  $E_{Rx} = -E_{Ix}$ . Further, the reflected electric field is perpendicular to the wave vector  $\vec{k}_R = \frac{k_R}{\sqrt{2}}(\hat{x} - \hat{z})$ . So the reflected electric field is written as

$$\vec{E}_R = \frac{E_{I0}}{\sqrt{2}} e^{i(k_x x - k_z z - \omega t)} (-\hat{x} - \hat{z})$$

At the boundary, the normal component of the electric field is not continuous, and the surface charge density is given by  $\sigma_f = \epsilon_0 \sqrt{2} E_{I0} e^{i(k_x x - \omega t)}$ . It is positive, since the normal component of the electric field at the boundary points away from the surface.

From Faraday's law, the reflected magnetic field is

$$\vec{B}_R = \frac{E_{I0}}{c} e^{i(k_x x - k_z z - \omega t)} \hat{y}$$

and is the same as the incident magnetic field at the surface. The field is tangential to the surface. So there is surface current at the boundary given by  $K_f = \frac{2E_{I0}}{\mu_0 c} e^{i(k_x x - \omega t)}$ , and its direction is  $\hat{x}$  according to Ampere's law (right-hand rule), since the magnetic field is in  $\hat{y}$  direction.

1(f) We take the wave solution of the electric field to the Maxwell equation to arrive at:

$$-k^2 + \mu\epsilon\omega^2 = -i\omega\mu\sigma$$

Using  $\sigma = i\sigma_0$  and with some algebra, we get the dispersion relation

$$k^2 = \mu\omega(\epsilon\omega - \sigma_0)$$

From this relation, if the solution of the wave number  $k$  has imaginary part, then the wave will be attenuated. That requires  $\epsilon\omega - \sigma_0 < 0$ , or  $\sigma_0 > \epsilon\omega$ . So low frequency waves will be attenuated. In this case,

$$k = \pm i\sqrt{\sigma_0 - \epsilon\omega}$$

and when  $k$  takes the positive sign (the forward direction), the wave will be damped other than propagating. In this case, since the magnetic field is related to the electric field by  $\vec{B} = \frac{1}{c}\vec{k} \times \vec{E}$ , there is a phase shift between  $\vec{B}$  and  $\vec{E}$  by  $\pi/2$ .

2(a) The magnetic dipole moment is given by  $\vec{m} = I\vec{a}$  where  $\vec{a}$  is the area vector of the loop coil. In this case,  $\vec{m} = \pi a^2 I \hat{z}$ .

2(b) To keep only first-order terms, the space derivative is only applied to the retarded term of the potential. Furthermore, the  $\vec{\nabla}$  operation on the retarded term results in  $-\frac{\hat{r}}{c}\frac{\partial}{\partial t}$ , so we get:

$$\vec{B} = \vec{\nabla} \times \vec{A} \approx \frac{\mu_0}{4\pi cr} \vec{\nabla} \times \left( \frac{\partial \vec{m}}{\partial t} \times \hat{r} \right) = \frac{\mu_0}{4\pi cr} \left( -\frac{1}{c} \hat{r} \right) \times \left( \frac{\partial^2 \vec{m}}{\partial t^2} \times \hat{r} \right) = \frac{\mu_0}{4\pi c^2 r} \hat{r} \times \left( \hat{r} \times \frac{\partial^2 \vec{m}}{\partial t^2} \right)$$

2(c) From (a), the direction of  $\vec{m}$  is  $\pm \hat{z}$ ; from (b), at the position of observer A,  $\hat{r} = \hat{y}$ , so the direction of the magnetic field is  $\hat{y} \times (\hat{y} \times \hat{z}) = \mp \hat{z}$ . The electric field direction can be determined from Faraday's law  $\vec{B} = \frac{1}{c}\vec{k} \times \vec{E}$ . In this case,  $\vec{k}$  is the wave direction which is  $\hat{y}$  at A's position, so it is seen that the direction of  $\vec{E}$  is  $\pm \hat{x}$ .

2(d) The radiation Poynting flux is in the direction of  $\hat{r}$ , so at A, the Poynting flux goes along the y-axis, and at B, the Poynting flux is along the line 45 degrees away from the y-axis in the yz plane. Solutions of  $\vec{E}$  and  $\vec{B}$  given in (b) show that both the electric and magnetic fields are proportional to  $\sin\theta/r$ , so the Poynting flux is proportional to  $\sin^2\theta/r^2$ , where  $\theta$  is the angle between the observer and  $\vec{m}$ , and  $r$  is the distance of the observer. In the given configurations,  $\theta_A = \pi/2$ ,  $\theta_B = \pi/4$ , and  $r_B = \sqrt{2}r_A$ , so  $S_A/S_B = 4$ .

3(a) The electric field given by one plate with surface charge density  $\sigma$  is  $E = \frac{\sigma}{2\epsilon_0}$  in the direction perpendicular to the plate and pointing from the positive charge to the negative charge. In the rest frame, the observer sees the charge density in both plates changed to  $\sigma_{\pm} = \sigma = \gamma\sigma_0$  because of the Lorentz contraction, with  $\gamma = 1/\sqrt{1-v^2/c^2}$ . So the electric field between the plates is twice the field by one plate, given by

$$\vec{E} = -\frac{\sigma}{\epsilon_0}\hat{y} = -\frac{\gamma\sigma_0}{\epsilon_0}\hat{y}$$

The magnetic field by a single plate is  $B = \pm \frac{\mu_0 K}{2}$ . In the rest frame, both plates have the same current density given by  $\vec{K} = \sigma v \hat{y}$  and in the same direction. From Ampere's law, the magnetic field in between becomes zero.

3(b) In this case, the charged particle receives the electric force by both plates and no magnetic force since it is at rest in its own rest frame. The electric field by the top plate is  $\vec{E}_+ = -\frac{\sigma_0}{2\epsilon_0}\hat{y}$ . The electric field by the bottom plate is  $\vec{E}_- = -\frac{\gamma_-\sigma_0}{2\epsilon_0}\hat{y}$  where  $\gamma_-$  is determined by the velocity of the bottom plate seen by the particle. We use Einstein's addition rule to find this velocity as

$$v_- = \frac{-v - v}{1 + v^2/c^2} = \frac{-2v}{1 + v^2/c^2}$$

and

$$\gamma_- = \frac{1}{\sqrt{1 - v_-^2/c^2}} = \frac{1 + v^2/c^2}{1 - v^2/c^2} = \gamma^2(1 + v^2/c^2)$$

So

$$\vec{E}_- = -\frac{\gamma^2(1 + v^2/c^2)\sigma_0}{2\epsilon_0}\hat{y}$$

and total electric field

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{-\gamma^2\sigma_0}{\epsilon_0}\hat{y}$$

The magnetic field is given by

$$\vec{B} = \vec{B}_- = \frac{1}{c^2}\vec{v}_- \times \vec{E}_- = \frac{\gamma^2 v \sigma_0}{\epsilon_0 c^2}(-\hat{x}) \times (-\hat{y}) = \gamma^2 \mu_0 v \sigma_0 \hat{z}$$

The charged particle only receives the electric force and no magnetic force at its own rest frame, and

$$\vec{F} = q\vec{E} = \frac{-q\gamma^2\sigma_0}{\epsilon_0}\hat{y}$$

3(c) In the particle's rest frame, the particle radiates with peak radiation in the direction perpendicular to the acceleration direction, or in the xz plane. In the lab's rest frame, the particle gives off synchrotron radiation with the peak radiation in the forward direction along the x-direction.

3(d) For the particle,  $\gamma_u = 1/\sqrt{1 - u^2/c^2} = \sqrt{2}$ . The total energy of particle is  $E = \gamma_u mc^2$ , and the relativistic momentum is  $\vec{p} = \gamma_u m(u_x, u_y, u_z) = \gamma_u m(0.5c, -0.5c, 0)$ , and the 4-element relativistic momentum is given by  $p^\mu = \sqrt{2}m(c, 0.5c, -0.5c, 0)$ .

3(e) Using Lorentz transformation, the first element of the 4-element relativistic momentum is transformed as  $p'^0 = \gamma p^0 - \gamma \beta p^1$ . Here  $\gamma = 1/\sqrt{1 - v^2/c^2}$ , and  $\beta = v/c$ . So  $p'^0 = \gamma\sqrt{2}(mc - v/c * 0.5mc) = \gamma\sqrt{2}(1 - 0.5v/c)mc$ , and the total energy is  $E' = p'^0 c = \gamma\sqrt{2}(1 - 0.5v/c)mc^2$ , and the kinetic energy is  $E'_k = E' - mc^2 = [\gamma\sqrt{2}(1 - 0.5v/c) - 1]mc^2$ . The total energy and kinetic energy in this frame are different from those measured in the lab frame, because the particle's velocity is different.