

Supplemental Notes

1. Currents

A current I is the amount of charges passing a cross-sectional area of a conducting wire per unit time. By definition, $I = \frac{dQ}{dt}$, in units of Ampere (A), and $1 \text{ A} = 1 \text{ C/s}$.

line current I : a line current I refers to the current in a line wire, in units of A. A (vector) line current may be envisioned as a line charge with a mean speed along the wire $\vec{I} = \lambda \vec{v}$.

surface current density K : the surface current density is the current per unit **length** (NOT area) perpendicular to the flow of the current. By definition, $K = \frac{dI}{da_{\perp}}$, in units of A/m. A (vector) surface current may be envisioned as surface charge density with a mean speed $\vec{K} = \sigma \vec{v}$.

volume current density J : the volume current density is the current per unit cross-sectional **area** (NOT volume) perpendicular to the flow of the current. By definition, $J = \frac{dI}{da_{\perp}}$, in units of A/m². A volume current density may be envisioned as volume charge density with a mean speed $\vec{J} = \rho \vec{v}$.

an area vector: An area vector \vec{a} is defined such that its magnitude is the area of the surface and its direction is the normal of the area, which is **perpendicular** to the plane of the surface and, by convention, pointing from inside the volume outward.

2. Waves in Conductors

With finite conductivity, the Maxwell's equations are reduced to the wave equation of the following form:

$$\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

If we describe the solution as $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$ so that the time and space arguments are in the exponential function, it can be easily seen that:

$$\begin{aligned} \nabla^2 &= -k^2 \\ \frac{\partial}{\partial t} &= -i\omega \\ \frac{\partial^2}{\partial t^2} &= -\omega^2 \end{aligned}$$

Therefore, the above wave equation is reduced to:

$$-k^2 + i\omega\mu\sigma + \mu\epsilon\omega^2 = 0$$

Since there is an imaginary term in the equation, the solution of the wave number must be a complex. We write this complex as $k = k_+ + ik_-$, and take it back to the equation:

$$-(k_+ + ik_-)^2 + i\omega\mu\sigma + \mu\epsilon\omega^2 = -k_+^2 + k_-^2 - 2ik_+k_- + i\omega\mu\sigma + \mu\epsilon\omega^2 = 0$$

To satisfy the equation, both the real part and imaginary part have to be satisfied, which leads to the other two equations:

$$k_+^2 - k_-^2 = \mu\epsilon\omega^2$$

$$2k_+k_- = \omega\mu\sigma$$

From the above second equation, $k_- = \frac{\omega\mu\sigma}{2k_+}$, which can be taken to the first equation to arrive at:

$$k_+^2 - \left(\frac{\omega\mu\sigma}{2k_+}\right)^2 - \mu\epsilon\omega^2 = 0$$

or

$$k_+^4 - \mu\epsilon\omega^2 k_+^2 - \frac{(\omega\mu\sigma)^2}{4} = 0$$

The solution of k_+^2 is therefore given by:

$$k_+^2 = \frac{\mu\epsilon\omega^2 \pm \sqrt{(\mu\epsilon\omega^2)^2 + (\omega\mu\sigma)^2}}{2}$$

Since k_+^2 is a positive real number, apparently, the “-” solution should be discarded. After some re-organization of terms, we arrive at:

$$k_+^2 = \mu\epsilon\omega^2 \frac{1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}}{2}$$

Or

$$k_+ = \omega\sqrt{\mu\epsilon} \sqrt{\frac{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1}{2}}$$

And k_- is further derived by:

$$k_-^2 = k_+^2 - \mu\epsilon\omega^2$$

leading to:

$$k_- = \omega\sqrt{\mu\epsilon} \sqrt{\frac{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1}{2}}$$

The plane wave solution in conductors can now be written as:

$$\vec{E} = \vec{E}_0 e^{-\vec{k}_- \cdot \vec{r}} e^{i(\vec{k}_+ \cdot \vec{r} - \omega t)}$$

3. EM Waves Between Conducting Plates

Suppose that a plane EM wave linearly polarized in x direction travelling in z direction between two parallel conducting plates which are parallel to the yz plane. One plate is placed at $x = 0$, the other plate is placed at $x = d$, and in between the plates is vacuum. We see that, between the plates, the following wave solution satisfies the Maxwell's equations:

$$\vec{E} = E_0 e^{i(kz - \omega t)} \hat{x}$$

$$\vec{B} = \frac{E_0}{c} e^{i(kz - \omega t)} \hat{y}$$

where c is the speed of light in vacuum. The wave is just like the plane wave in unbounded vacuum, except that at $x > d$ and $x < 0$, the fields vanish, since we assume that the conducting plates are perfect conductors. In this case, at the boundary with the two plates, the boundary condition has to be satisfied. We note that, at the boundary $x = 0, x = d$, the electric field in our solution only has the normal component, and the magnetic field only has the tangential component. Both fields vanish, or are discontinuous at the boundary, suggesting that surface charge σ_f and surface current \vec{K}_f are present. Below we will find the surface charge and surface current distribution at the conducting surface.

The boundary condition for the electric field is $(\epsilon_{out} \vec{E}_{out} - \epsilon_{in} \vec{E}_{in}) \cdot \hat{n} = \sigma_f$ and for the magnetic field is $\vec{B}_{out}/\mu_{out} - \vec{B}_{in}/\mu_{in} = \vec{K}_f \times \hat{n}$ where “out” in this case refers to the plates, “in” refers to the vacuum between the plates, and \hat{n} is the normal of the boundary surface pointing from inside outward. Apparently, $\vec{E}_{out} = \vec{0}$, and $\vec{B}_{out} = \vec{0}$ as the fields in the conducting plates vanish.

We first find the charge distribution. At the top boundary $x = d$, we see that $\hat{n} = \hat{x}$, so the boundary condition for the electric field yields $(\epsilon_{out} \vec{E}_{out} - \epsilon_{in} \vec{E}_{in}) \cdot \hat{n} = -\epsilon_0 E_0 e^{i(kz - \omega t)} \hat{x} \cdot \hat{x} = -\epsilon_0 E_0 e^{i(kz - \omega t)} = \sigma_f$, leading to

$$\sigma_f(x = d) = -\epsilon_0 E_0 e^{i(kz - \omega t)}$$

At the bottom boundary $x = 0$, $\hat{n} = -\hat{x}$, the boundary condition leads to $(\epsilon_{out} \vec{E}_{out} - \epsilon_{in} \vec{E}_{in}) \cdot \hat{n} = -\epsilon_{in} E_0 e^{i(kz - \omega t)} \hat{x} \cdot (-\hat{x}) = \epsilon_0 E_0 e^{i(kz - \omega t)} = \sigma_f$, so that

$$\sigma_f(x = 0) = \epsilon_0 E_0 e^{i(kz - \omega t)}$$

We therefore see the electric field and charge distribution as if the electric field be generated by surface charges on a parallel capacitor, with the electric field pointing from the positive charge plate to the negative charge plate. However, different from the case of electrostatic field by charges of a capacitor, in this case, both the electric field and surface charges are varying with time (t) and space (z): they are traveling fields and charges.

We then examine the surface current distribution using the boundary condition for the magnetic field $\vec{B}_{out}/\mu_{out} - \vec{B}_{in}/\mu_{in} = \vec{K}_f \times \hat{n}$. With some algebra, this equation may be further written as $\hat{n} \times (\vec{B}_{out}/\mu_{out} - \vec{B}_{in})/m\mu_{in} = \hat{n} \times (\vec{K}_f \times \hat{n}) = \vec{K}_f (\hat{n} \cdot \hat{n}) - \hat{n} (\vec{K}_f \cdot \hat{n})$. In this configuration, the

surface current is in the plate parallel to the yz plane, so $\vec{K}_f \cdot \hat{n} = 0$, and the above relation is reduced to

$$\hat{n} \times (\vec{B}_{out}/\mu_{out} - \vec{B}_{in}/\mu_{in}) = \vec{K}_f$$

At the top boundary $x = d$, $\hat{n} = \hat{x}$, and the boundary given above yields

$$\vec{K}_f(x = d) = \hat{x} \times (-\vec{B}_{in})/\mu_0 = -\frac{E_0}{\mu_0 c} e^{i(kz - \omega t)} \hat{z}$$

At the bottom boundary $x = 0$, $\hat{n} = -\hat{x}$, so the boundary condition yields

$$\vec{K}_f(x = 0) = -\hat{x} \times (-\vec{B}_{in})/\mu_0 = \frac{E_0}{\mu_0 c} e^{i(kz - \omega t)} \hat{z}$$

The configuration appears as if the magnetic field be produced by current running in $-z$ direction along the upper plates and returning in z direction along the lower plate. However, different from the case of magnetostatic field generated by steady current, in this case, the surface current varies in time and space, just as the magnetic field, both of which travel along z direction.

4. EM Waves in Co-axial Cables

Now if we consider to roll the parallel conducting plates in the above example into a cable transmission line made of long co-axial cylindrical shells with the inner radius a and outer radius b . In between the shells is vacuum. Again, an EM wave travels along the axis (z direction) of the cable between the shells. It is straight forward to conceive that the electromagnetic field configuration be the same as the case between conducting plates, with the electric field pointing from one shell to the other, and the magnetic field being azimuthal, or tangential to the cable surface. Therefore, the wave solution may be written as

$$\begin{aligned} \vec{E} &= E(s) e^{i(kz - \omega t)} \hat{s} \\ \vec{B} &= \frac{E(s)}{c} e^{i(kz - \omega t)} \hat{\phi} \end{aligned}$$

Here, different from the case in Cartesian coordinates, E and B are functions of s , which may be determined by Gauss's law and Faraday's law. Given the cylindrical symmetry, we consider that the fields are not functions of the azimuthal angle ϕ . The Gauss's law for the field in vacuum requires $\vec{\nabla} \cdot \vec{E} = 0$. In cylindrical coordinates, this is written as

$$\frac{1}{s} \frac{\partial s E(s)}{\partial s} = 0$$

Solving the differential equation, we get the solution $E(s) = \frac{A}{s}$, where A is a constant to be determined later. With Faraday's law, we can further determine the magnetic field. Finally, the fields are written as

$$\begin{aligned} \vec{E} &= \frac{A}{s} e^{i(kz - \omega t)} \hat{s} \\ \vec{B} &= \frac{A}{cs} e^{i(kz - \omega t)} \hat{\phi} \end{aligned}$$

Now let's take a detour to determine the constant A . The radial electric field between the shells indicates an electric potential, or voltage difference $V = V_a - V_b$, between the two shells. We will find this potential difference by

$$V = V_a - V_b = - \int_b^a \vec{E} \cdot d\vec{s} = \int_a^b \vec{E} \cdot \hat{s} ds = \int_a^b \frac{A}{s} ds e^{i(kz-\omega t)} = A \ln\left(\frac{b}{a}\right) e^{i(kz-\omega t)}$$

The potential difference is also a wave solution which we further write as $V \equiv V_0 e^{i(kz-\omega t)}$, with V_0 being the amplitude of an alternating voltage between the two shells. We therefore derive the constant A as expressed by V_0 , and the fields are now written as

$$\begin{aligned} \vec{E} &= \frac{V_0}{\ln(b/a)} \frac{1}{s} e^{i(kz-\omega t)} \hat{s} \\ \vec{B} &= \frac{V_0}{c \ln(b/a)} \frac{1}{s} e^{i(kz-\omega t)} \hat{\phi} \end{aligned}$$

With these fields, we will examine the surface charge and current distribution on the inner and outer shells. Here let's define that "out" refers to $s > b$ or $s < a$, and "in" refers to the vacuum between the shells where the fields reside. So $\vec{E}_{out} = \vec{0}$, and $\vec{B}_{out} = \vec{0}$ as the fields vanish in perfect conducting shells.

At the outer boundary $s = b$, $\hat{n} = \hat{s}$, and the boundary condition for the electric field is written as $(\epsilon_{out} \vec{E}_{out} - \epsilon_{in} \vec{E}_{in}) \cdot \hat{n} = -\epsilon_0 E(s) e^{i(kz-\omega t)} \hat{s} \cdot \hat{s} = \sigma_f$, leading to

$$\sigma_f(s = b) = -\frac{\epsilon_0 V_0}{b \ln(b/a)} e^{i(kz-\omega t)}$$

At the inner boundary $s = a$, $\hat{n} = -\hat{s}$, and the boundary condition gives

$$\sigma_f(s = a) = \frac{\epsilon_0 V_0}{a \ln(b/a)} e^{i(kz-\omega t)}$$

For the surface current, we examine the boundary condition for the magnetic field, which can be re-written as $\vec{K}_f = \frac{1}{\mu_0} \vec{B}_{in} \times \hat{n}$. At the outer boundary $s = b$, $\hat{n} = \hat{s}$, we therefore derive the surface current

$$\vec{K}_f(s = b) = \frac{-V_0}{\mu_0 c b \ln(b/a)} e^{i(kz-\omega t)} \hat{z}$$

At the inner boundary, $s = a$, $\hat{n} = -\hat{s}$, and the surface current is derived as

$$\vec{K}_f(s = a) = \frac{V_0}{\mu_0 c a \ln(b/a)} e^{i(kz-\omega t)} \hat{z}$$

The electric field appears like electrostatic field by charges carried by the co-axial cable, and the magnetic field appears like magnetostatic field by currents carried by the cable. However, they are **not** static fields, they are self-feeding EM waves traveling along the cable, and the charge and

current distribution also travel along the cable. In this way, the transmission line conveys message from one end of the cable to the other end by either fields or charges and currents.

It is found that the total current on the inner or outer shell is

$$I(s = a) = K(s = a)2\pi a = \frac{2\pi V_0}{\mu_0 c \ln(b/a)} e^{i(kz - \omega t)} = \frac{2\pi V_0 \epsilon_0 c}{\ln(b/a)} e^{i(kz - \omega t)}$$

$$I(s = b) = K(s = b)2\pi b = \frac{-2\pi V_0}{\mu_0 c \ln(b/a)} e^{i(kz - \omega t)} = \frac{-2\pi V_0 \epsilon_0 c}{\ln(b/a)} e^{i(kz - \omega t)}$$

So the inner and outer shells carry currents of the same magnitude but opposite signs. It indicates an alternating current related to an alternating voltage $V = V_0 e^{i(kz - \omega t)}$ by

$$Z = \frac{V}{I} = \frac{\ln(b/a)}{2\pi \epsilon_0 c}$$

If a linear material (ϵ and $\mu \approx \mu_0$) fills in between the cables, then we get

$$Z = \frac{\ln(b/a)}{2\pi \epsilon v} \approx \sqrt{\frac{\epsilon_0}{\epsilon}} \frac{\ln(b/a)}{2\pi \epsilon_0 c}$$

We define Z as impedance, which is a property only determined by the geometry and the material of the cable line. Note that the impedance has exactly the same units as the resistance. However, it is not resistance. The transmission line in this discussion does not have resistance, but currents, charges, and fields travel along the cable by the self-feeding EM wave.

5. Radiation by Point Electric Dipole

A time-varying electric dipole $\vec{p}(t)$ at the origin is made by two time-varying point charges of $\pm q(t)$ placed at $\pm d/2$ on the z axis. This setup generates the retarded scalar potential as: $V(\vec{r}, t) = V_+ + V_-$, where

$$V_+ = \frac{1}{4\pi\epsilon_0} \frac{q(t - \iota_+/c)}{\iota_+}$$

$$V_- = \frac{1}{4\pi\epsilon_0} \frac{-q(t - \iota_-/c)}{\iota_-}$$

where V_+ and V_- are produced by the positive and negative charges at the distance ι_+ and ι_- respectively. Now expanding ι using the assumption $d \ll r$, we get:

$$\iota_{\pm} = r(1 \mp \frac{d}{2r} \cos \theta)$$

and

$$\frac{1}{\iota_{\pm}} = \frac{1}{r} (1 \pm \frac{d}{2r} \cos \theta)$$

We further expand the retarded term using the ι from above:

$$q(t - \iota_{\pm}/c) = q(t - r/c) \pm \frac{d}{2c} \cos \theta \frac{\partial q(t - r/c)}{\partial t}$$

The above expansion only keeps the first order term by ignoring the higher-order time derivatives of q , the underlying assumption being

$$\frac{d}{2c} \cos \theta \frac{\partial}{\partial t} \ll 1$$

or $d \ll c/\omega$. Taking all these terms back to the retarded potential calculation, and ignoring the higher order term involving $1/r^2$ by assuming $r \gg c/\omega$, we get:

$$V = \frac{1}{4\pi\epsilon_0 c} \frac{q(\tau) d \cos \theta}{r}$$

Here we define $\tau \equiv t - r/c$. Now in the setup, the electric dipole moment is given as $\vec{p} = qd\hat{z}$. So the above can be further written as:

$$V = \frac{1}{4\pi\epsilon_0 c} \frac{1}{r} \hat{r} \cdot \frac{\partial \vec{p}}{\partial t}$$

In the similar manner, by discarding higher order terms, the vector potential is given by

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{1}{r} \frac{\partial \vec{p}}{\partial t}$$

To derive the radiation electric and magnetic fields, the key is to keep only derivatives applied to retarded term which has an implicit r dependence, but discard derivatives applied to explicit r term. Specifically, there are:

$$\vec{\nabla} = -\frac{1}{c} \hat{r} \frac{\partial}{\partial t}$$

$$\begin{aligned}\vec{\nabla} \cdot &= -\frac{1}{c} \hat{r} \cdot \frac{\partial}{\partial t} \\ \vec{\nabla} \times &= -\frac{1}{c} \hat{r} \times \frac{\partial}{\partial t}\end{aligned}$$

The radiation electric field and magnetic field are given by:

$$\begin{aligned}\vec{E} &= \frac{\mu_0}{4\pi} \frac{1}{r} \left[\hat{r} \times \left(\hat{r} \times \frac{\partial^2 \vec{p}}{\partial t^2} \right) \right] \\ \vec{B} &= -\frac{\mu_0}{4\pi c} \frac{1}{r} \left(\hat{r} \times \frac{\partial^2 \vec{p}}{\partial t^2} \right) = \frac{1}{c} \hat{r} \times \vec{E}\end{aligned}$$

Note that in these solutions, $\vec{p} \equiv \vec{p}(\tau)$, where $\tau \equiv t - r/c$. And these are general solutions for radiations fields by a point electric dipole located at the origin, independent of the coordinate system.

6. Vector Potential by a Circular Current Ring

A circular current ring of radius b placed in the xy plane and centered at the origin makes a magnetic dipole. The magnetic dipole moment is given by $\vec{m} = I\pi b^2 \hat{z}$. Now if the current varies with time as $I = I_0 \cos(\omega t)$, the dipole radiates. We may calculate the retarded vector potential of the point dipole at an arbitrary location \vec{r} by

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l}'}{\iota} = \frac{\mu_0 I_0 b}{4\pi} \int \frac{\cos[\omega(t - \iota/c)] d\phi' \hat{\phi}}{\iota}$$

It is seen that $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z}$, $\vec{r}' = b\cos\phi'\hat{x} + b\sin\phi'\hat{y}$, and

$$\iota = (r^2 + r'^2 - 2\vec{r} \cdot \vec{r}')^{1/2} = r \left[1 + \frac{b^2}{r^2} - 2\frac{b}{r} \sin\theta \cos(\phi - \phi') \right]^{1/2} \approx r \left[1 - \frac{b}{r} \sin\theta \cos(\phi - \phi') \right]$$

And with the far-field approximations with all higher order terms ommitted (see textbook), the integral becomes

$$\vec{A} = -\frac{\mu_0 I_0 \omega b^2}{4\pi r c} \sin(\omega\tau) \sin\theta \int \cos(\phi - \phi') d\phi' \hat{\phi}$$

where $\tau \equiv t - r/c$. Note that the integral should carry along in the Cartesian coordinates since $\hat{\phi}$ varies. We further write $\hat{\phi} = -\sin\phi'\hat{x} + \cos\phi'\hat{y}$, and the integral becomes:

$$\vec{A} = -\frac{\mu_0 I_0 \omega b^2}{4\pi r c} \sin(\omega\tau) \sin\theta \int \cos(\phi - \phi') (-\sin\phi'\hat{x} + \cos\phi'\hat{y}) d\phi'$$

The above integral for \hat{x} and \hat{y} components results in

$$\vec{A} = -\frac{\mu_0 I_0 \omega \pi b^2}{4\pi r c} \sin(\omega\tau) \sin\theta (-\sin\phi\hat{x} + \cos\phi\hat{y}) = -\frac{\mu_0 I_0 \omega \pi b^2}{4\pi r c} \sin(\omega\tau) \sin\theta \hat{\phi}$$

as given in the textbook (Equation 11.35).

Further, with the notion that $\vec{m} = m_0 \cos(\omega\tau) \hat{z}$, $\frac{\partial m}{\partial t} = -\omega \sin(\omega\tau) m_0$, and $\sin\theta \hat{\phi} = -\hat{r} \times \hat{z}$, the retarded vector potential can be further written into its general form independent of the coordinate system as

$$\vec{A} = -\frac{\mu_0}{4\pi} \frac{1}{cr} (\hat{r} \times \frac{\partial \vec{m}}{\partial t})$$

The electric field is given by

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi} \frac{1}{rc} (\hat{r} \times \frac{\partial^2 \vec{m}}{\partial t^2})$$

and the magnetic field is given by

$$\vec{B} = \vec{\nabla} \times \vec{A} \approx \frac{\mu_0}{4\pi} \frac{1}{c^2 r} [\hat{r} \times (\hat{r} \times \frac{\partial^2 \vec{m}}{\partial t^2})] = \frac{1}{c} \hat{r} \times \vec{E}$$

Above are general solutions for radiations fields by a point magnetic dipole located at the origin, independent of the coordinate system.

7. Radiation by Point Dipoles

To summarize, the radiation by electric dipole placed at origin is given by:

$$\begin{aligned} \vec{E} &= \frac{\mu_0}{4\pi} \frac{1}{r} [\hat{r} \times (\hat{r} \times \frac{\partial^2 \vec{p}}{\partial t^2})] = -c\hat{r} \times \vec{B} \\ \vec{B} &= -\frac{\mu_0}{4\pi c} \frac{1}{r} (\hat{r} \times \frac{\partial^2 \vec{p}}{\partial t^2}) = \frac{1}{c} \hat{r} \times \vec{E} \\ \vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0 c} \vec{E} \cdot \vec{E} \hat{r} = \frac{\mu_0}{16\pi c} \frac{1}{r^2} |\hat{r} \times \frac{\partial^2 \vec{p}}{\partial t^2}|^2 \hat{r} \end{aligned}$$

The radiation by magnetic dipole placed at origin is given by:

$$\begin{aligned} \vec{E} &= \frac{\mu_0}{4\pi c} \frac{1}{r} (\hat{r} \times \frac{\partial^2 \vec{m}}{\partial t^2}) = -c\hat{r} \times \vec{B} \\ \vec{B} &= \frac{\mu_0}{4\pi c^2} \frac{1}{r} [\hat{r} \times (\hat{r} \times \frac{\partial^2 \vec{m}}{\partial t^2})] = \frac{1}{c} \hat{r} \times \vec{E} \\ \vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0 c} \vec{E} \cdot \vec{E} \hat{r} = \frac{\mu_0}{16\pi c^3} \frac{1}{r^2} |\hat{r} \times \frac{\partial^2 \vec{m}}{\partial t^2}|^2 \hat{r} \end{aligned}$$

The above solutions are independent of the coordinate system. It is seen that the electric field and magnetic field always oscillate in the spherical surface centered at the dipole and are orthogonal to each other. The Poynting flux always points from the origin to the observer in the radial direction, and it is orthogonal to the electric field and magnetic field.

In terms of the magnitude, the magnitude of the electric field and magnetic field is related to the position of the observer: it is proportional to $\sin \psi$ with ψ being the angle between the observer

and the direction of the dipole moment, and inversely proportional to the observer's distance. The Poynting flux is proportional to $\sin^2 \psi$ and inversely proportional to distance squared. In the direction along the dipole moment, the observer detects no radiation, and in the plane perpendicular to the dipole moment, the observer detects the strongest radiation.

Furthermore, comparing the magnetic dipole radiation with the electric dipole radiation, it is seen that for a source with both, the magnetic dipole radiation is a lot smaller than the electric dipole radiation, hence is often ignored (see textbook). In your assignment (Griffiths Problem 11.9), only radiation by the electric dipole moment is to be computed.

8. Radiation by Relativistic Particles

The formula describing radiation by relativistic particles are given in textbook Example 11.3 and Problem 11.16. Note that the angle θ in the book refers to the observer's position with respect to the velocity β of the charge. In the case of collinear acceleration, θ is also the angle with the acceleration, but in the case of synchrotron radiation, θ is NOT the angle with acceleration. When discussing the angular dependence, make sure what θ refers to. In the case of collinear acceleration, radiation near the forward direction is strongest; in the case of synchrotron radiation, radiation in the forward direction (and perpendicular to the acceleration) is strongest. In comparison, slow-moving charges radiate most strongly in the direction perpendicular to the acceleration, and the radiation power beams are symmetric about the \vec{a} axis.

The total radiation power by collinear particles is

$$P_{\parallel} = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}$$

and the total radiation power by particles with $\vec{a} \perp \vec{\beta}$ is

$$P_{\perp} = \frac{\mu_0 q^2 a^2 \gamma^4}{6\pi c}$$

. If charged particles are accelerated by electromagnetic forces, then $a \sim 1/m$, so $P \sim 1/m^2$. Therefore, in particle acceleration experiments, light particles radiate a lot more than heavy particles. For example, an electron radiates a lot more than a proton by more than a million times.

In magnetic fields, charged particles have a circular motion, and the acceleration is by the Lorentz force as $\gamma m a = qvB$. Therefore, the total radiation power is given by

$$P = \frac{\mu_0 q^4 \gamma^2 v^2 B^2}{6\pi c m^2}$$

. Again, it is seen that light particles (e.g. electrons) radiate a lot more than heavy particles (e.g. protons), and the radiation power is proportional to B^2 as well as $v^2 \gamma^2$. For relativistic particles, $v^2 \gamma^2$ is a big number.

A relativistic electron has the total energy $E = \gamma m_e c^2$, where $\gamma = 1/\sqrt{1 - \beta^2}$ is the Lorentz factor, and m_e is the electron rest mass. This energy includes the kinetic energy E_k and the rest mass energy $m_e c^2$ of the electron, therefore, $E_k = (\gamma - 1)m_e c^2$, leading to $\gamma = 1/\sqrt{1 - \beta^2} = E'_k + 1$, where $E'_k = E_k/M_e c^2$ is the kinetic energy relative to the rest mass energy. The rest mass energy of an electron is given by $m_e c^2 = 0.91 \times 10^{-30} \times (3 \times 10^8)^2 \text{J} = 0.91 \times 10^{-30} \times (3 \times 10^8)^2 / 1.6 \times 10^{-19} \text{eV} = 512 \text{keV}$. So for $E_k = 100 \text{keV}$, $\gamma = (100/512.) + 1 = 1.20$, and $\beta = 0.55$. It can be found that the synchrotron radiation power by an electron of 100 keV kinetic energy for $B = 10^8 \text{T}$ is

$$P = \frac{\mu_0 q^4 \gamma^2 v^2 B^2}{6\pi c m^2} = \frac{\mu_0 c q^4 \gamma^2 \beta^2 B^2}{6\pi m^2} \approx 70 \text{J/s}$$

which is a large amount of power. Even in weak magnetic fields $B \sim 1 \text{T}$, the radiation power by a 100 keV electron is about 40 keV per second.