

## Solutions to Quiz 5

(a) Radiation by moving charges is proportional to  $a^2$ . For charges accelerated by the electromagnetic force, there is  $\gamma m \vec{a} = q(\vec{E} + \vec{v} \times \vec{B})$ , or  $a \propto 1/m$ , and  $a^2 \propto 1/m^2$ . Therefore, the light particles radiate more. Between protons and electrons, since  $m_p \approx 2000m_e$ , an electron radiates much more than a proton by roughly  $2000^2 \approx 10^6$ , or by a factor of a few million.

Does the velocity of the charge also make a difference? Yes it does. In this situation, the velocity enters the power as proportional to  $\gamma^2$  or  $\gamma^4$  (why?). The difference by  $\gamma$  is not as prominent as  $m^2$  in general. So order-of-magnitudely, we mostly take into account the  $1/m^2$  effect.

(b) When a particle crosses the gap, it is accelerated by the electric field, and its acceleration is along the direction of its velocity, or collinear acceleration with  $\vec{a} \parallel \vec{\beta}$ . In this case, the radiation power beam is given by

$$\frac{dP_{||}}{d\Omega} \propto \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

where  $\theta$  is the angle between the observer's direction and particle's velocity or acceleration. For very slow particles,  $\beta \approx 0$ , the peak radiation is at  $\theta = \pi/2$ , so the power beam is perpendicular to and symmetric about the particle trajectory. In a few cycles as the particle's velocity grows, the power beam is leaning forward, where forward is defined as the direction of the particle's velocity. As  $v \sim c$ , the peak radiation almost lies along the forward direction, but not exactly on the forward direction. This picture may be achieved by sampling a few special angles such as  $\theta_0$ ,  $\theta = \pi/2$ ,  $\theta = \pi$ .

(c) Inside the Ds,  $\vec{a} \perp \vec{\beta}$ , the particle gives off synchrotron radiation with the peak radiation in the forward direction, i.e.,  $\cos \theta = 1$ , when the  $1/(1 - \beta \cos \theta)^5$  is the most prominent. Therefore, as the particle follows a circular motion, the peak radiation beam also changes direction to be always aligned with the particle's velocity.

(d) We first find the acceleration of the particle by the magnetic force as  $\gamma m a = qvB$ , leading to  $a = qvB/(\gamma m)$ . So the total synchrotron radiation power is given by

$$P_{\perp} = \frac{\mu_0 q^4 B^2 v^2 \gamma^2}{6\pi c m^2}$$

Further, the time it takes for the particle to finish half a circle is

$$\Delta t = \frac{1}{2}T = \frac{\pi \gamma m}{qB}$$

So the total radiation energy by the particle is

$$W = P_{\perp} \Delta t = \frac{\mu_0 q^3 B v^2 \gamma^3}{6cm}$$

And compared with its kinetic energy

$$\frac{W}{mc^2} = \frac{\mu_0 q^3 B v^2 \gamma^3}{6c^3 m^2}$$

For particle radiation energy to be comparable with its kinetic energy, the magnetic field strength would be

$$B \sim \frac{6c^3 m^2}{\mu_0 q^3 v^2 \gamma^3}$$

In this case,  $\gamma = E_k/(mc^2) + 1 = 2$ , and  $v^2/c^2 = 1 - 1/\gamma^2 = 3/4$ , so the above becomes

$$B \sim \frac{6c^3 m^2}{\mu_0 q^3 v^2 \gamma^3} = \frac{6cm^2 \times 4}{\mu_0 q^3 2^3 \times 3} \approx 10^{11} \text{T}$$

This is very strong magnetic field only found in magnetars.

(e) The radiation force is proportional to  $\partial \vec{a}/\partial t$ ; so as long as the acceleration vector changes with time, there is radiative damping. During the collinear acceleration when the particle is accelerated by the electric field, if the electric field is not constant, or even if  $E$  is constant but the particle mass  $\gamma m$  is changing with its velocity,  $\partial \vec{a}/\partial t$  is not zero, and its direction is always along the particle trajectory. During the synchrotron radiation,  $\vec{a}$  changes direction (as well as magnitude if relativistic effect is taken into account). In the case of small  $\beta$  when  $\gamma \approx 1$ ,  $\partial \vec{a}/\partial t \parallel -\vec{\beta}$ , so the radiative force is in the opposite direction of particle's velocity. This makes sense that radiative damping would try to slow down the particle or reduce the kinetic energy of the particle.