

Solution to HW Problems

38. (a) Note that the electric field generated by a large plate is a constant, normal to the plate, and changes sign across the plate, as $E = \pm \frac{\sigma}{2\epsilon_0}$. The electric field goes from the plate outward if the surface charge is positive. In this problem, the electric field is the superposition by two plates of opposite charges, so the field in between the plates is not zero and twice the field generated by one plate $E = \frac{\sigma}{\epsilon_0}$, pointing from the positive charge to the negative charge, while the electric field outside the plates is zero. The electric energy is therefore found to be

$$W = \int u_E d\tau = \frac{\epsilon_0}{2} \int E^2 d\tau = \frac{\epsilon_0}{2} \left(\frac{\sigma}{\epsilon_0}\right)^2 \int d\tau = \frac{\epsilon_0}{2} \left(\frac{\sigma}{\epsilon_0}\right)^2 Ad = \frac{Q^2 d}{2\epsilon_0 A}$$

(b) The two plates with charges of opposite signs exert an attractive force on each other. The force by one plate on the other is a constant regardless the distance between the two plates and is normal to the plates, which is given by

$$F = fA = \frac{\sigma^2}{2\epsilon_0} A = \frac{\sigma^2}{2\epsilon_0} A = \frac{Q^2}{2\epsilon_0 A}$$

And the work done by an external force to overcome the electrostatic force to move one plate away from the other by a distance d is therefore

$$W = Fd = \frac{Q^2 d}{2\epsilon_0 A}$$

(c) Here let's assume the electric potential on the positive plate as V_+ and the potential of the negative plate as V_- . We may just choose the plate of the negative charge as the reference point, so that $V_- = V_{ref}$. We then compute

$$V_+ = V_{ref} - \int_-^+ \vec{E} \cdot d\vec{l} = V_- + \frac{\sigma}{\epsilon_0} d$$

The energy of the negative plate is

$$W_- = \frac{1}{2} \int -\sigma V_- da = -\frac{1}{2} \sigma V_- A$$

, and the energy of the positive plate is

$$W_+ = \frac{1}{2} \int \sigma V_+ da = \frac{1}{2} \sigma (V_- + \frac{\sigma}{\epsilon_0} d) A$$

So the total energy is

$$W = W_+ + W_- = \frac{1}{2} \frac{\sigma^2}{\epsilon_0} Ad = \frac{Q^2 d}{2\epsilon_0 A}$$

Note that here we seem to have ignored the cross term, but indeed we did not, because V_- and V_+ are the electric potential by both plates.

(d) The work done to charge the capacitor can be given by

$$W = \int dW = \int_0^Q \frac{q dq}{C} = \frac{1}{2} \frac{Q^2}{C}$$

From the class notes, we find that the capacitance of the parallel plates capacitor is $C = \epsilon_0 A/d$, so

$$w = \frac{Q^2 d}{2\epsilon_0 A}$$

39. (a) This is a problem of solving a 1D Laplace equation in Cartesian coordinates. Since the plates are in xy plane, the electrostatic potential is only a function of z , so the Laplace's equation becomes:

$$\frac{\partial^2 V}{\partial z^2} = 0$$

The solution is simply $V(z) = az + b$. We use the boundary condition to find a and b . At $z = 0$, $V = b = V_0$, and at $z = d$, $\frac{\partial V}{\partial z} = a = -E_z = -\frac{\sigma}{\epsilon_0}$. Therefore,

$$V = -\frac{\sigma}{\epsilon_0} z + V_0$$

(b) In this problem, we solve a 1D Laplace's equation in spherical coordinates. The electrostatic potential only depends on r , so the Laplace's equation becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} = 0$$

Solving this equation, we get the general solution $V = \frac{a}{r} + b$. The constants a and b can be determined using the boundary condition. At $r = \infty$, $V(\infty) = \frac{a}{\infty} + b = 0$, so $b = 0$. At $r = R$, the boundary condition is

$$E(R) = \frac{\sigma}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 R^2}$$

On the other hand,

$$E(R) = -\frac{\partial V}{\partial r} \Big|_{r=R} = \frac{a}{R^2}$$

, so we get

$$a = \frac{Q}{4\pi\epsilon_0}$$

Therefore, the solution of V is

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

This is what we are familiar with.

40. Griffiths 3.13 If the strip at $x = 0$ is a conductor with constant electrostatic potential, then inside the conductor, the electric field is zero, and all charges are on the surface of $x = 0$. The charge density satisfies the boundary condition $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0} \hat{n}$. Here we take \vec{E}_{above} as electric field at $x \geq 0$, and $\vec{E}_{below} = 0$ being the electric field inside the conductor. The normal of the surface is simply $\hat{n} = \hat{x}$. Therefore, we shall find the x component of the electric field at the boundary $x = 0$. We may directly use the result of Equation 3.36, which gives the electrostatic potential. The electric field is derived as $\vec{E} = -\vec{\nabla}V$, so

$$E_x = -\frac{\partial V}{\partial x} = -\frac{4V_0}{\pi} \sum_{n=1,3,5..} \frac{-n\pi}{a} \frac{1}{n} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) = \frac{4V_0}{a} \sum_{n=1,3,5..} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$$

At the boundary $x = 0$, we get

$$E_x(0, y) = \frac{4V_0}{a} \sum_{n=1,3,5..} \sin\left(\frac{n\pi y}{a}\right)$$

So the surface charge density is

$$\sigma = \epsilon_0 E_x(0, y) = \frac{4\epsilon_0 V_0}{a} \sum_{n=1,3,5..} \sin\left(\frac{n\pi y}{a}\right)$$

41. Griffiths 3.14 In this problem, we first determine the form of the solution by directly using the boundary conditions. First, the pipe in the z -dimension runs from $-\infty$ to ∞ , or there is the translational symmetry in z -dimension, so the electrostatic potential is independent of z and is only a function of x and y : $V \equiv V(x, y) = V_x(x)V_y(y)$. Second, at $y = 0$ and $y = a$, $V = 0$, so the form of V_y must be a sine function which can achieve zero at two boundaries, i.e., $V_y(y) \sim \sin(k_y y)$. Further, to satisfy that $V_y(y = a) = 0$, $k_y = n\pi/a$, $n = 1, 2, 3, \dots$, so $V_y \sim \sin(n\pi y/a)$. Third, the boundary condition at $V(0, y) = 0$ at $x = 0$ requires that V_x can be only a sine or hyperbolic sine function $V_x(x) \sim \sin(k_x x)$ or $V_x(x) \sim \sinh(k_x x)$. The 2D Laplace's equation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ also requires that $C_x + C_y = 0$, where $C_x = \frac{\partial^2 V_x}{\partial x^2}$, and $C_y = \frac{\partial^2 V_y}{\partial y^2}$. Since $C_y = -k_y^2 < 0$, we must have $C_x = k_x^2 = k_y^2 > 0$, so the form of V_x must be $V_x \sim \sinh(k_x x)$ and $k_x = k_y = n\pi/a$, leading to $V_x \sim \sinh(n\pi x/a)$. We therefore find a solution $V_n = V_x V_y = C_n \sinh(n\pi x/a) \sin(n\pi y/a)$ for any positive integer $n = 1, 2, 3, \dots$

Note that any linear combination of the above solutions also satisfies the Laplace's equation, so we write the general solution as $V = \sum_{n=1,2,3..} C_n \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$, and the coefficients C_n can be determined using the last boundary condition $V(x = b, y) = V_0$. Here again we use the Fourier analysis tricks to find that

$$\int_0^a \sum_{n=1,2,3..} C_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = C_n \sinh\left(\frac{n\pi b}{a}\right) \frac{a}{2}$$

only when $n' = n$, so we get

$$C_n \sinh\left(\frac{n\pi b}{a}\right) \frac{a}{2} = \int_0^a \sum_{n=1,2,3..} C_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) dy = \int_0^a V_0 \sin\left(\frac{n\pi y}{a}\right) dy = \frac{2V_0 a}{n\pi}$$

only when $n = 1, 3, 5, \dots$. Organizing the above terms, we get

$$C_n = \frac{4V_0}{n\pi \sinh(\frac{n\pi b}{a})}$$

So the final solution is

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\dots} \frac{\sinh(\frac{n\pi x}{a}) \sin(\frac{n\pi y}{a})}{n \sinh(\frac{n\pi b}{a})}$$