

Part III: Electrodynamics (Chap7)

Review: now that we've seen and used all four Maxwell's equations describing electrostatics and magnetostatics:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \text{ (Gauss's law)}$$

$$\text{(in matter: } \nabla \cdot \vec{D} = \rho_f \text{)}$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \text{ (Ampere's law)}$$

$$\text{(in matter, } \nabla \times \vec{H} = \vec{J}_f \text{)}$$

In configurations of symmetry, Gauss's law may be used to find \vec{E} or \vec{D} , and Ampere's law may be used to find \vec{B} or \vec{H} .

In general configurations, we introduce electric potential V and magnetic vector potential \vec{A} , such that $\vec{E} = -\nabla V$ (satisfying $\nabla \times \vec{E} = 0$) and $\vec{B} = \nabla \times \vec{A}$ (satisfying $\nabla \cdot \vec{B} = 0$). V and \vec{A} may be derived by solving Poisson's or Laplace's equations and matching solutions to the boundary conditions.

An electric potential difference is the amount of work the electric field done on unit charge. Such cannot be extended to \vec{A} (because magnetic force does NO work). In magnetostatics, \vec{A} does not seem as useful as V in electrostatics (i.e., do NOT try to find \vec{A} in order to determine \vec{B} unless you do not have any other means, such as Ampere's law or approximation to a dipole field. Note that a magnetic dipole field is the second term, or the first significant term, in the multipole expansion to find \vec{A}).

Polarization or magnetization of materials is often described using the electric dipole \vec{p} (may be viewed as a pair of opposite charges) or magnetic dipole \vec{m} (may be viewed as a loop current). The field generated by polarized/magnetized materials is equivalent to the field generated by bound charges or bound currents. Only surface charges are present in a uniformly polarized conductor, and only surface currents are present in a uniformly magnetized conductor.

In linear media, $\vec{P} = \chi_e \epsilon_0 \vec{E}$, and $\vec{M} = \chi_m \vec{H}$; and, $\vec{D} = \epsilon \vec{E}$, and $\vec{H} = \mu \vec{B}$.

III.1 Ohm's law

An electric field in a conductor drives charges into motion, producing an electric current. Empirically, the current density is scaled to the electric field by $\vec{J} = \sigma \vec{E}$, where σ is the conductivity of the conductor. This is the local Ohm's law.

A steady current running along a conductor is driven by a maintained potential difference, or $I = V/R$, where R is the resistance of the conductor and is determined by the conductivity as well

as the geometry of the conductor.

Review: find electric potential and electric field in given configurations using Gauss's law.

Example 1: resistance of a cylinder conductor (see book).

Example 2: resistance of a conductor connecting two co-axial cylindrical shells (see book).

Application: conductivity of metals and insulators.

Example 3: the time variation of charge, potential difference, and electric current when a charged capacitor starts to discharge.

In the above example (also see HW problem # 24), the electrostatic potential energy original stored in the capacitor is converted, by the electrostatic field doing work on charges, into *Joule heating* when the capacitor is connected to a resistor to discharge. The power of Joule heating is $P = VI = I^2R = V^2/R$.

III.2 Electromotive force

In the very general form, a current may be driven by many kinds of forces such that $\vec{J} = \sigma \vec{f}$, where \vec{f} is the force per unit charge. Electrostatic force is but one special case. In order to maintain a current in a circuit, such as by hooking the resistor to a battery, the integral of the force along the circuit is introduced as the electromotive force, or $emf \epsilon$. At times, ϵ is equivalent to the work done on unit charge throughout the circuit. Per this definition, emf by the electrostatic force is zero. In other words, the electrostatic field does no net work on charges in a circuit.

Cautions on emf : 1. emf is NOT a force, but the integral of the force per unit charge; 2. we are most concerned with emf as integral along the circuit, i.e., the path of a loop (current); 3. it is the integral along the circuit, NOT along the actual displacement of charges.

Like in the case of a battery, you might envision emf as the work done by a certain force so as to maintain the potential difference between the two electrodes of the battery, which is converted into Joule heating. Therefore, $\epsilon = V$, $I = \epsilon/R$, and the power of Joule heating is $P = \epsilon I = \epsilon^2/R$.

For our purpose, we will only look at emf by electromagnetic forces.

Example 4: motional emf : a moving wire in the magnetic field. Find out what is the force driving the electric current, what is the emf , what is the work done, and what agent does the work.

Flux rule: a motional emf is equivalent to the change rate of magnetic flux enclosed by the loop circuit $\epsilon = -\frac{d\Phi}{dt}$.

The change of magnetic flux through a circuit may be achieved by (1) moving the wire, or changing the shape of the circuit in a constant magnetic field; (2) "moving" the magnetic field (such as moving a bar magnet close to or away from a circuit) and (3) changing the magnetic field. However, the emf is by the *magnetic force* in (1) and *electric force* in (2) and (3).

III.3 Induction

Faraday’s law: a changing magnetic field induces an electric field: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Example 5: static versus induced electric fields. An electrostatic field is generated by static electric charges (the divergence of the electric field) by Gauss’s law; an induced electric field is generated by changing magnetic field (the curl of the electric field) by Faraday’s law. Both electric fields exert electric forces on charges by $\vec{F} = Q\vec{E}$, and the local Ohm’s law applies to both fields: $\vec{J} = \sigma\vec{E}$. The loop path integral of an electrostatic field is zero, but the loop path integral of an induced electric field is the induced EMF.

Example 6: induced electric field and current in a loop circuit next to a moving bar magnet (see book): using flux rule and Lenz’s law to find the induced electric current.

The integral form of Faraday’s law may be used to find the induced electric field in configurations of symmetry, just like how the magnetostatic field may be found using the integral form of Ampere’s law. The Amperian loops for various configurations can be applied to Faraday’s law by making analogy to Ampere’s law. To do so, you may replace the $\mu_0\vec{J}$ term in Ampere’s law by $-\partial\vec{B}/\partial t$ term in Faraday’s law, or replace the $\mu_0 I_{enc}$ term by $-\partial\psi_{enc}/\partial t$ term.

The first-level problem: you may be asked to find induced \vec{E} , induced EMF, and induced current given the form of $\vec{B}(t)$ using Faraday’s law. The second-level problem: you may be asked to find induced \vec{E} , induced EMF, and induced current given a time varying current distribution from which you first should find \vec{B} using Ampere’s law. Be cautious not to confuse the “given” current with the induced current.

Example 7: the induced electric field, EMF, and current by a time varying uniform \vec{B} confined in a region or in the entire space.

Example 8: the induced electric field, EMF, and current by a solenoid with time varying electric current. While the magnetic field is uniform inside and zero outside, the induced electric field fills the entire space. The electric field is azimuthal, its magnitude increasing from the axis of the solenoid until the surface of the solenoid, and then decreasing with distance. The maximum induced electric field is at the surface of the solenoid.

Example 9: the induced electric field, EMF, and current in various configurations of a solenoid. E.g., a solenoid with a magnetic material filling various parts of the cylinder, and a solenoid with a magnetically conducting bar being inserted into the cylinder, a

Note that although an electric field is induced by a solenoid carrying a time varying current, it does not automatically give rise to appreciable induced current unless in electrically conducting materials, i.e., $\vec{J} = \sigma\vec{E}$ where $\sigma \neq 0$.

Note that to find the induced EMF and current in a conducting loop wire, you directly use the flux rule and Lenz’s law and do not have to find induced \vec{E}_{in} first.

Example 10: a betatron accelerator: accelerating electrons by induced electric field. Different from inside a cyclotron, electrons accelerated in a betatron torus of a fixed radius. The betatron is designed so that the magnetic field at the radius of the torus the one half of the mean magnetic field across the area of the torus.

III.4 Inductance

Mutual inductance: a time varying current I_1 in Loop 1 induces an EMF ϵ_2 and current I_2 in Loop 2 by $\epsilon_2 = -M_{21} \frac{dI_1}{dt}$. M_{21} is called the mutual inductance, in units of Henry = V m / A. It can be proven that $M_{12} = M_{21} = M$, which is only determined by the geometry and relative orientation and position of the two loops or loop systems, independent of the electro-magnetic field and currents.

To find the mutual inductance, you may find the magnetic field \vec{B} produced by an imaginary current I in one loop, and the magnetic flux $\Phi = \int \vec{B} \cdot \vec{da}$ through the second loop, and $M = \Phi/I$. Theoretically, you would reach the same answer no matter which loop you choose to run an imaginary current (because $M_{12} = M_{21} = M$). Practically, you would choose to calculate \vec{B} from the loop of a geometry that is easier for the calculation.

Example 10: find the mutual inductance between two solenoids of various configurations.

Example 11: find the induced current along a long wire by a time varying current in a small square loop next to the wire.

Self inductance: the magnetic flux Φ through a loop by a current I running in the loop is given by $\Phi = LI$, where L is the self-inductance of the loop. L is an intrinsic property only determined by its geometry. A time varying current I in the loop induces an EMF ϵ_{in} in itself by $\epsilon_{in} = -d\Phi/dt = -L \frac{dI}{dt}$. ϵ_{in} always opposes the change of I in the loop, so is called the *back EMF*.

Example 12: find the self inductance of a long solenoid.

Example 13: find the self inductance of a long conductor of coaxial cylindrical shell.

Example: 14: find the self-inductance of a toroidal coil.

Example 15: a LR circuit. Because of the back EMF, when a current is switched on in a circuit, it takes a certain time (the time constant of the LR circuit) to reach the desired value.

Example 16: induced EMF in an AC circuit and an AC transformer. The example shows that, through the self-induction in the primary and second coils as well as the mutual induction between the two coils, the source energy is temporarily stored as current energy in the inductors (or the magnetic energy) and eventually converted into Joule heating.

III.5 Magnetic energy

When a current is built up from zero to a certain value, the work done against the back EMF is the energy stored in the current. Therefore, $W = \frac{1}{2}LI^2$. This is the energy of the magnetic field in

the entire space $W = \int \frac{1}{2\mu_0} B^2 d\tau$, which is equivalent to the work done against the induced electric field when building up the field.

The self-inductance may be derived by calculating the magnetic energy.

Example 17: the magnetic energy in a solenoid.

Example 18: the magnetic energy in a long cylindrical-shell cable line and the self-inductance of the cable.

III.6 Maxwell Equations

Ampere's law only describes the magnetostatic field, or the magnetic field by a steady current. A changing electric field induces a magnetic field. This is described by the displacement current $\vec{J}_d = \epsilon_0 \partial \vec{E} / \partial t$.

Example 19: the induced magnetic field by the displacement current in a charging capacitor.

Example 20: the induced electric field and magnetic field by a time-varying current.

III.6 Maxwell's equations and boundary conditions

Now we've got all 4 Maxwell's equations to describe the time-varying electric field and magnetic field:

$$\nabla \cdot \vec{E} = \rho / \epsilon_0 \text{ (Gauss's law)}$$

$$\text{(in matter: } \nabla \cdot \vec{D} = \rho_f \text{)}$$

$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t \text{ (Faraday's law)}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \partial \vec{E} / \partial t \text{ (Ampere-Maxwell's law)}$$

$$\text{(in matter, } \nabla \times \vec{H} = \vec{J}_f + \partial \vec{D} / \partial t \text{)}$$

In linear media, $\vec{D} = \epsilon \vec{E}$, and $\vec{H} = \vec{B} / \mu$.

Maxwell's equations describe the relationship between fields, and between fields and sources (charges and currents).

The boundary conditions are such that the normal component of the magnetic field is always continuous and the tangential component of the electric field is always continuous. The discontinuity in the normal component of the electric field is determined by the surface charge and permittivities of the two media across the boundary. The discontinuity of the tangential component of the magnetic field is determined by the surface current and permeabilities of the two media across the boundary.