

Part IV: Maxwell's Equations and EM waves (Chap 9)

IV.1 Maxwell's equations and EM waves in free space

The four Maxwell's equations describe the relationship between the magnetic field and the electric field as well as the relationship between fields and sources (charges and currents). In addition, the Lorentz's force law $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ describes the electric-magnetic forces on charges, and the Ohm's law $\vec{J} = \sigma \vec{E}$ describes the relationship between currents and electric fields in conductors.

In vacuum where there are no free charges and currents, Maxwell's equations reduce to a wave equation for the electric field as well as the magnetic field:

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \partial^2 \vec{E} / \partial t^2$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \partial^2 \vec{B} / \partial t^2$$

where the wave speed $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light in vacuum. Therefore, light is EM waves.

In a linear media (ϵ, μ) with no free charges and currents, we also obtain wave equations, and the phase speed of waves becomes $v = 1/\sqrt{\mu \epsilon} < c$.

IV.2 Plane waves

Example 1: deriving the solutions of monochromatic plane waves by separating variables.

One simple wave solution in unbounded free space is a sinusoidal monochromatic plane wave, often written as $\vec{E}(\vec{r}) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$. The wave solution is given in a complex form for mathematic simplicity, but the physical quantity is the real part of the solution. \vec{E}_0 is a complex constant vector, giving the amplitude and phase shift of the wave.

In this solution, the electric field vector varies in space and time. \vec{k} is the wave vector, its direction indicating the wave propagation direction, and its magnitude being the wave number $k = 2\pi/\lambda$. ω is the wave frequency. k and ω are related by $v = \omega/k$.

For symmetry between the electric field and magnetic field in the Maxwell's equation, the wave solution of the magnetic field is exactly like that of the electric field. But \vec{E} and \vec{B} are not independent of each other. \vec{B} may be derived from \vec{E} using Faraday's law.

Example 2: deriving the solution of \vec{B} from Faraday's law.

Example 3: properties of monochromatic plane waves in vacuum: (a) plane waves are transverse waves, with \vec{E} and \vec{B} both perpendicular to the wave propagation direction, and perpendicular to each other: $\vec{E} \cdot \vec{k} = 0, \vec{B} \cdot \vec{k} = 0, \vec{B} = \vec{k} \times \vec{E} / \omega$. (b) EM waves transport electromagnetic energy, the energy flux (Poynting flux) being $\vec{S} = \vec{E} \times \vec{B} / \mu_0$. In vacuum, the energy of the electric field equals the energy of the magnetic field. (c) wave polarization: linearly polarized waves have \vec{E} always oscillating in the same direction (e.g., along x-axis), in circularly polarized waves, the \vec{E} vector rotates about the z-axis (if waves propagate along the z-axis). Such is also true with the magnetic field, but the \vec{B} vector is always perpendicular to \vec{E} .

IV.3 EM waves in matter and in conductors

In unbounded space of matter, EM plane waves propagate with a reduced velocity $v = 1/\sqrt{\epsilon\mu}$. The ratio $n = c/v = ck/\omega$ is called the refraction index of the media. Waves travelling from one media to another are reflected and transmitted.

Example 4: reflection and transmission of normal incident plane waves: beyond the boundary, there are three plane waves, the incident wave, reflected wave, and transmitted wave. At the charge/current free boundary, the boundary condition has to be satisfied: the normal and tangential components of both \vec{E} and \vec{B} are continuous across the boundary.

In conductors where currents are present by Ohm's law $\vec{J} = \sigma\vec{E}$, the solution of the Maxwell's equations becomes $\vec{E} = \vec{E}_0 e^{-k_2 z} e^{i(k_1 z - \omega t)}$. k_1 and k_2 are real positive numbers. The $e^{-k_2 z}$ term indicates the wave diminishing with the distance, and the length scale $\delta = 1/k_2$ is defined as the "skin depth" to characterize penetration of EM waves in conductors.

Example 5: EM waves in perfect conductors: $k_2 = \infty, \delta = 0, |\vec{E}| = |\vec{B}| = 0$. EM waves propagating to conductor are completely reflected.

IV.4 transmission lines

A transmission line is a pipe with conducting walls, which transports along its length signals in the form of either \vec{E} and \vec{B} , or charges and currents. In the ideal situation (perfect conducting walls), there is no loss of signal and energy along the line.

Example 6: EM plane waves between two large conducting plates. EM waves are the same as in free unbounded space, but the electric field induces charges and currents at the conducting walls, which may be derived from the boundary conditions.

Example 7: EM waves in a coaxial cable and impedance. Note that if the cable is made of perfect conductor, the resistance of the cable is zero, and there is no field, but currents flow in the cable (think about the Ohm's law).

Note that in the ideal transmission line, the electric and magnetic fields are identical to electrostatic and magnetostatic fields by charges and currents at the walls. \vec{E} and \vec{B} are exact plane wave solutions of the Maxwell's equations. It can be proved that the tangential component of the electric field and the normal component of the magnetic field are zero everywhere in the transmission line.