

## Solutions to Homework Problem Set 17

Reif 6.1

(a) The ratio of electron number at the first excited state to that at the ground state is given by

$$\frac{N_1}{N_0} = e^{\frac{-(\epsilon_1 - \epsilon_0)}{kT}} = e^{\frac{-(3/2\hbar\omega - 1/2\hbar\omega)}{kT}} = e^{\frac{-\hbar\omega}{kT}}$$

When  $\hbar\omega \gg kT$ ,  $N_1/N_0 \approx 0$ , or almost all electrons stay at the ground state.

(b) The mean energy of this two-state system is

$$\epsilon_{mean} = \frac{\epsilon_0 P_0 + \epsilon_1 P_1}{P_0 + P_1} = \frac{\frac{1}{2}\hbar\omega e^{-\frac{1}{2}\frac{\hbar\omega}{kT}} + \frac{3}{2}\hbar\omega e^{-\frac{3}{2}\frac{\hbar\omega}{kT}}}{e^{-\frac{1}{2}\frac{\hbar\omega}{kT}} + e^{-\frac{3}{2}\frac{\hbar\omega}{kT}}} = \frac{1}{2}\hbar\omega \frac{3 + e^{\frac{\hbar\omega}{kT}}}{1 + e^{\frac{\hbar\omega}{kT}}}$$

When  $\hbar\omega \gg kT$ , the mean energy is nearly  $\frac{1}{2}\hbar\omega$ , or the ground-state energy.

Stowe 15.6

(a) By normalizing the probability distribution  $\sum P_n = C \sum e^{-n \times 0.02eV/(kT)} = 1$ , we derive  $C = (\sum e^{-n \times 0.02eV/(kT)})^{-1}$ . Here the sum is done over all integers of  $n$  from 0 to infinity. Let  $x = e^{-0.02eV/(kT)}$ , then  $C = (1 + x + x^2 + x^3 \dots + x^n + \dots)^{-1}$ . Since  $-0.02eV/kT < 0$ , we get  $x < 1$ , then the sum  $1 + x + x^2 + x^3 \dots + x^n + \dots$  is the Taylor expansion of the expression  $1/(1-x)$ . Therefore,  $C = (1-x) = 1 - e^{-0.02eV/(kT)} = 1 - e^{-0.02 \times 1.602 \times 10^{-19} / (1.381 \times 10^{-23} \times 273.)} = 0.573$ .

(b) The probability of the system in the ground state is  $P_0 = C e^0 = C = 0.573$ .

(c) The probability of the system in the first excited state is  $P_1 = C e^{-0.02eV/kT} = 0.245$ .

Stowe 15.17

(a) The relative number of electrons in a certain state is scaled by the Boltzmann factor  $N_\epsilon \sim e^{-\epsilon/kT}$ . So the ratio of the number of electrons  $N_0$  at the ground state with energy  $\epsilon_0$  to the number of electrons  $N_1$  at the first excited state with energy  $\epsilon_1$  is given by

$$\frac{N_0}{N_1} = \frac{e^{-\epsilon_0/kT}}{e^{-\epsilon_1/kT}} = e^{-(\epsilon_0 - \epsilon_1)/kT}$$

Given that  $N_0/N_1 = 10$ ,  $\epsilon_1 - \epsilon_0 = 0.2eV$  (note that the excited states have higher energy than the ground state), we can derive the equilibrium temperature as

$$T = \frac{-(\epsilon_0 - \epsilon_1)}{k \ln \frac{N_0}{N_1}} = \frac{0.2eV \times 1.6 \times 10^{-19} J/eV}{1.38 \times 10^{-23} J/K \times \ln 10} \approx 1000K$$

(b) The excitation temperature is defined such that

$$\frac{\epsilon_1 - \epsilon_0}{kT} = 1$$

So

$$T = \frac{\epsilon_1 - \epsilon_0}{k} = \frac{0.2\text{eV} \times 1.6 \times 10^{-19}\text{J/eV}}{1.38 \times 10^{-23}\text{J/K}} \approx 2300\text{K}$$

At this temperature,  $N_1/N_0 = 1/e \approx 0.37$ .