

Solutions to Homework Problem Set 20

Stowe 16.8

The speed distribution is such $P(v)dv = 4\pi v^2 \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} dv$. Here $P(v)$ is the probability density, and $P(v)dv$ is the probability in the speed range of v and $v + dv$. At the maximum of the distribution, $\partial P(v)/\partial v = 0$, leading to $2ve^{-\frac{mv^2}{2kT}} - \frac{2mv^2}{2kT} e^{-\frac{mv^2}{2kT}} = 0$, or $v = \sqrt{\frac{2kT}{m}}$. This speed at the maximum distribution is the most probable speed.

Stowe 16.11

(a) The average velocity is zero, because there is equal chance for particles to move in either direction.

(b) The average speed is derived using the speed distribution by $\bar{v} = \int_0^\infty vP(v)dv = \int_0^\infty 4\pi v^3 e^{-\frac{mv^2}{2kT}} dv = \sqrt{\frac{8kT}{\pi m}} = 4.7 \times 10^2$ m/s.

(c) The root mean square speed may be derived from the mean kinetic energy, since $\bar{v}^2 = 2\bar{\epsilon}/m$. For a gas, the mean kinetic energy of the translational motion is $\bar{\epsilon} = \frac{3}{2}kT$, so $\bar{v}^2 = \frac{3kT}{m}$, and $v_{rms} = \sqrt{\bar{v}^2} = \sqrt{3kT/m} = 5.1 \times 10^2$ m/s.

(d) The most probable speed is $v_{max} = \sqrt{2kT/m} = 4.2 \times 10^2$ m/s.

These speeds are of order the sound speed at room temperature. And it is seen that $v_{rms} > \bar{v} > v_{max}$.

Reif 7.22

In this problem, we use the Maxwell velocity distribution, as only one component of the velocity (the so-called line-of-sight component) causes the Doppler shift. We treat this component as v_x , and use the $P(v_x)dv_x = \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{mv_x^2}{2kT}} dv_x$ to find the spread of Doppler-shift.

(a) The mean frequency is derived by $\bar{\nu} = \int_{-\infty}^\infty \nu P(v_x)dv_x = \int_{-\infty}^\infty \nu_0 \left(1 + \frac{v_x}{c}\right) P(v_x)dv_x = \int_{-\infty}^\infty \nu_0 P(v_x)dv_x + \int_{-\infty}^\infty \nu_0 \frac{v_x}{c} P(v_x)dv_x$. The first term of the integral is $\nu_0 \int_{-\infty}^\infty P(v_x)dv_x = \nu_0$, and the second term of the integral is proportional to the mean value of v_x , which is zero; i.e., $\frac{\nu_0}{c} \int_{-\infty}^\infty v_x P(v_x)dv_x = \frac{\nu_0}{c} \bar{v}_x = 0$. Therefore, the mean frequency $\bar{\nu} = \nu_0$. So the line is only broadened but the line center is not shifted.

(b) Using the result from (a), it is seen that $\nu - \bar{\nu} = \nu - \nu_0 = \nu_0 \frac{v_x}{c}$. So we can find the mean value of $(\nu - \bar{\nu})^2$ to be $\int_{-\infty}^\infty \nu_0^2 \left(\frac{v_x}{c}\right)^2 P(v_x)dv_x = \left(\frac{\nu_0}{c}\right)^2 \left(\frac{m}{2\pi kT}\right)^{1/2} \int_{-\infty}^\infty v_x^2 e^{-\frac{mv_x^2}{2kT}} dv_x = \left(\frac{\nu_0}{c}\right)^2 \frac{kT}{m}$. So the root-mean-square width is $(\Delta\nu)_{rms} = \frac{\nu_0}{c} \left(\frac{kT}{m}\right)^{1/2}$.

Note that here you can directly get \bar{v}_x^2 by the fact that $\bar{v}_x^2 = 2\bar{\epsilon}_x/m$, where $\bar{\epsilon}_x$ is the mean kinetic energy of translational motion in X-direction. According to equipartition, $\bar{\epsilon}_x = kT/2$, so $\bar{v}_x^2 = kT/m$.

(c) Since ν is proportional to v_x , so $I(\nu)$ distribution is basically the same as v_x distribution.

Therefore, $I(\nu)d\nu = I_0 e^{-\frac{mv_x^2}{2kT}} d\nu$, where I_0 is the intensity at the line center when $\nu = \nu_0$ or $v_x = 0$.
From the relation between ν and v_x , we get $v_x = \frac{c}{\nu_0}(\nu - \nu_0)$. Therefore, $I(\nu)d\nu = I_0 e^{-\frac{mc^2(\nu - \nu_0)^2}{2kT\nu_0^2}} d\nu$.