Solutions to Homework Problem Set 24

Stowe 22.5

(a) Note that the Einstein model is a simplified model which assumes that all $3N_a$ oscillators have the same characteristic frequency $\omega_0$. Therefore, all the oscillators have the same energy and the same occupation number. The total thermal energy is therefore $E = \Sigma_{n=1}^{3N_a} \bar{n}_s \epsilon_n = \Sigma_{n=1}^{3N_a} \epsilon_0 / (e^{\epsilon_0/kT} - 1) = 3N_a \epsilon_0 / (e^{\epsilon_0/kT} - 1)$.

(b) The temperature at which the thermal energy is approximated by the classical value (equipartition) is when $kT \approx \epsilon_0$. Given $T = 100$ K, then $\epsilon_0 = kT = 1.38 \times 10^{-23} \text{J/K} \times 100 \text{K} = 1.38 \times 10^{-21}$ J. And the characteristic frequency is $\omega_0 = \epsilon_0 / \hbar = 1.3 \times 10^{13}$ Hz.

(c) $E/3N_a kT = x / (e^x - 1)$.

(d) The function $E/3N_a kT = x / (e^x - 1)$ decreases with increasing $x$. When $x \sim 0$, or the high temperature limit, $x / (e^x - 1) \approx x / (1 + x - 1) = 1$, so the thermal energy is $3N_a kT$. When $x$ approaches infinity, or the low temperature limit, $x / (e^x - 1) \approx e^{-x}$, which approaches zero.

Stowe 22.18

(a) In this problem, if we are going to treat the liquid helium with the Debye model, we may consider that 1 mole of liquid helium as $3N_a$ coupled oscillators with $3N_a$ characteristic frequencies. In the same way Debye determined the maximum frequency, we may use the condition $3N_a = \int_{\epsilon_m}^\infty \epsilon g(\epsilon) d\epsilon$. Here $\epsilon_m = h\nu_m$, where $\nu_m$ is the Debye frequency. For liquid, since only the longitudinal wave can propagate, so $g(\epsilon)$ is different from that for the solids by a factor of 3. This way, the integral thus yields $3N_a = \frac{4\pi V h^3 c^3 s}{3\nu_m^3 \epsilon_D^3}$. Or $\epsilon_D = 0.62hc_s(\frac{N_a}{M})^{1/3} \times 3^{1/3}$, and $\Theta_D = \epsilon_D / k$. $V$ for 1 mole of liquid helium is $V = 6.64 \times 10^{-3} \text{kg/mol} / 145 \text{kg/m}^3 = 4.58 \times 10^{-5} \text{m}^3 / \text{mol}$, and $C_s = 223 \text{m/s}$, so $\Theta_D \approx 24.1 \text{K}$.

(b) We find the total thermal energy by the integral $E = \int_{\epsilon_m}^\infty \epsilon n(\epsilon) g(\epsilon) d\epsilon$, where $g(\epsilon)$ is smaller than that for solids by a factor of 3. The integral yields $E = \frac{4\pi V}{h c_s^2} \int_0^{\epsilon_m} \epsilon^3 e^{-\epsilon/kT} d\epsilon$. The term $\frac{4\pi V}{h c_s^2}$ is replaced by $\frac{9N_a}{\epsilon_D}$ using the integral result in (a). At low temperatures, the molar specific heat is derived by

$$c_V = \frac{\partial E}{\partial T} = \frac{12N_a \pi^4 k}{5} \left( \frac{T}{\Theta_D} \right)^3 \times \frac{1}{6.64 \times 10^{-3} \text{kg/mol} J / \text{K/mol}}$$

With $\Theta_D$ calculated above, we get $c_V = 20.9T^3 \text{J/mol/K}^4$, which is very similar to the measured value.