

## Solutions to Homework Problem Set 24

### Stowe 22.5

(a) Note that the Einstein model is a simplified model which assumes that all  $3N_a$  oscillators have the same characteristic frequency  $\omega_0$ . Therefore, all the oscillators have the same energy and the same occupation number. The total thermal energy is therefore  $E = \sum_{s=1}^{3N_a} \bar{n}_s \epsilon_s = \sum_{s=1}^{3N_a} \epsilon_0 / (e^{\epsilon_0/kT} - 1) = 3N_a \epsilon_0 / (e^{\epsilon_0/kT} - 1)$ .

(b) The temperature at which the thermal energy is approximated by the classical value (equipartition) is when  $kT \approx \epsilon_0$ . Given  $T = 100$  K, then  $\epsilon_0 = kT = 1.38 \times 10^{-23} \text{ J/K} \times 100 \text{ K} = 1.38 \times 10^{-21}$  J. And the characteristic frequency is  $\omega_0 = \epsilon_0/\hbar = 1.3 \times 10^{13}$  Hz.

(c)  $E/3N_a kT = x/(e^x - 1)$ .

(d) The function  $E/3N_a kT = x/(e^x - 1)$  decreases with increasing  $x$ . When  $x \sim 0$ , or the high temperature limit,  $x/(e^x - 1) \approx x/(1 + x - 1) = 1$ , so the thermal energy is  $3N_a kT$ . When  $x$  approaches infinity, or the low temperature limit,  $x/(e^x - 1) \approx e^{-x}$ , which approaches zero.

### Stowe 22.18

(a) In this problem, if we are going to treat the liquid helium with the Debye model, we may consider that 1 mole of liquid helium as  $3N_a$  coupled oscillators with  $3N_a$  characteristic frequencies. In the same way Debye determined the maximum frequency, we may use the condition  $3N_a = \int_0^{\epsilon_m} g(\epsilon) d\epsilon$ . Here  $\epsilon_m = h\nu_m$ , where  $\nu_m$  is the Debye frequency. For liquid, since only the longitudinal wave can propagate, so  $g(\epsilon)$  is different from that for the solids by a factor of 3. This way, the integral thus yields  $3N_a = \frac{4\pi V}{3h^3 c_s^3} \epsilon_D^3$ . Or  $\epsilon_D = 0.62hc_s (\frac{N_a}{V})^{1/3} \times 3^{1/3}$ , and  $\Theta_D = \epsilon_D/k$ .  $V$  for 1 mole of liquid helium is  $V = 6.64 \times 10^{-3} \text{ kg/mol} / 145 \text{ kg/m}^3 = 4.58 \times 10^{-5} \text{ m}^3/\text{mol}$ , and  $C_s = 223 \text{ m/s}$ , so  $\Theta_D \approx 24.1 \text{ K}$ .

(b) We find the total thermal energy by the integral  $E = \int_0^{\epsilon_m} \epsilon n(\epsilon) g(\epsilon) d\epsilon$ , where  $g(\epsilon)$  is smaller than that for solids by a factor of 3. The integral yields  $E = \frac{4\pi V}{h^3 c_s^3} \int_0^{\epsilon_D} \frac{\epsilon^3 d\epsilon}{e^{\epsilon/kT} - 1}$ . The term  $\frac{4\pi V}{h^3 c_s^3}$  is replaced by  $\frac{9N_a}{\epsilon_D^3}$  using the integral result in (a). At low temperatures, the molar specific heat is derived by

$$c_V = \partial E / \partial T = \frac{12N_a \pi^4 k}{5} \left( \frac{T}{\Theta_D} \right)^3 \times \frac{1}{6.64 \times 10^{-3} \text{ kg/mol}} \text{ J/K/mol}$$

With  $\Theta_D$  calculated above, we get  $c_V = 20.9T^3 \text{ J/mol/K}^4$ , which is very similar to the measured value.