

Solutions to Homework Problem Set 6

Stowe 7.31

(a) The total number of states in the combined system is the multiplet of the numbers of states in sub-systems, therefore, $\Omega_0 = \Omega_1\Omega_2 = 10^{5 \times 10^{24}}$.

(b) By definition, $S = k \ln \Omega$, so $S_1 = k \ln \Omega_1 = k \times 2 \times 10^{24} \times \ln 10 = 63.6$ J/K, and $S_2 = k \ln \Omega_2 = k \times 3 \times 10^{24} \times \ln 10 = 95.3$ J/K. Note that the entropy S has the same units (J/K) as the Boltzmann constant k .

(c) The total entropy of the combined system is $S_0 = S_1 + S_2 = 159$ J/K. Or, the total entropy can be also derived as $S_0 = k \ln \Omega_0 = k \times 5 \times 10^{24} \times \ln 10 = 159$ J/K.

Stowe 7.37

(a) Using equipartition theorem that the mean energy per degree of freedom is $\epsilon = \frac{1}{2}kT$, the total internal energy of the ideal gas, which is also the thermal energy of the gas, is given by $E = \frac{1}{2}kTf$, where f is the number of degrees of freedom. For a monatomic ideal gas, each gas particle has 3 degrees of freedom (translational motions), so $f = 3N$, and $E = \frac{1}{2}3NkT = 6.2 \times 10^5$ J. Note that here the absolute temperature $T = 273 + 27 = 300$ K has to be used.

(b) The number of states of a gas is given by $\Omega \propto E^{\frac{f}{2}} \propto T^{\frac{f}{2}}$. When the temperature is raised by $1^\circ \text{C} = 1^\circ \text{K}$, or $T' = T + 1$, the number of states will change to $\Omega' \propto T'^{\frac{f}{2}}$. The number of states will increase because the internal energy of the gas increases with the temperature. The factor by which the number of states increases is derived by

$$\frac{\Omega'}{\Omega} = (T'/T)^{\frac{f}{2}} = \left(\frac{301}{300}\right)^{\frac{f}{2}} = 1.0033^{\frac{f}{2}} = e^{\frac{f}{2} \ln(1.0033)}$$

We apply the Taylor series expansion to the above result to arrive at:

$$\frac{\Omega'}{\Omega} \approx e^{\frac{0.0033f}{2}} = e^{5 \times 10^{23}} = 10^{5 \times 10^{23} \log(e)} = 10^{2.2 \times 10^{23}}$$

(c) By definition, the entropy $S = k \ln \Omega = C + \frac{1}{2}fk \ln E = C + \frac{1}{2}fk \ln\left(\frac{fkT}{2}\right)$, where C is a constant including the terms not related to E . So, the change in entropy can be written as $\Delta S = S' - S = \frac{1}{2}fk \ln\left(\frac{fkT'}{2}\right) - \frac{1}{2}fk \ln\left(\frac{fkT}{2}\right) = \frac{1}{2}fk \ln\left(\frac{T'}{T}\right)$. Since the temperature increases by a small amount compared with the original temperature, we use Taylor series expansion to the above result to arrive at: $\Delta S \approx \frac{1}{2}fk \frac{(T'-T)}{T} = 6.9$ J/K.

Stowe 8.3

(a) From the expression of the number of states $\Omega \propto E^{\frac{f}{2}}$, it is seen that A_1 and A_2 have 20 and 16 degrees of freedom, respectively.

(b) The number of states of the combined system $\Omega_0 = \Omega_1\Omega_2$. In the final equilibrium, Ω_0 should be maximized, or equivalently, we may maximize $\ln \Omega_0 = \ln \Omega_1 + \ln \Omega_2$. $\ln \Omega_0$ is a function of

independent variable E_1 , while E_2 is dependent on E_1 by the energy conservation $E_2 = E_0 - E_1$. To maximize $\ln\Omega_0$, its derivative with respect to all independent variables should be zero. In this case, $\frac{\partial \ln\Omega_0}{\partial E_1} = 0$. From the expression of Ω_0 , there is:

$$\frac{\partial \ln\Omega_0}{\partial E_1} = \frac{\partial \ln\Omega_1}{\partial E_1} + \frac{\partial \ln\Omega_2}{\partial E_2} \frac{\partial E_2}{\partial E_1} = \frac{\partial \ln\Omega_1}{\partial E_1} - \frac{\partial \ln\Omega_2}{\partial E_2} = \frac{10}{E_1} - \frac{8}{E_2}$$

Letting $\frac{\partial \ln\Omega_0}{\partial E_1} = 0$ leads to $\frac{10}{E_1} - \frac{8}{E_2} = 0$, or $\frac{E_1}{10} = \frac{E_2}{8} = \frac{E_1 + E_2}{18}$. So $E_1 = \frac{10}{18}E_0 = 0.56 \times 10^{-18}$ J, and $E_2 = \frac{8}{18}E_0 = 0.44 \times 10^{-18}$ J.

(c) The entropy of the combined system in final equilibrium is $S_0 = S_1 + S_2$ when E_1 and E_2 are taking the values given in (b). Therefore, $S_0 = k(\ln\Omega_1 + \ln\Omega_2) = 10k\ln E_1 + 8k\ln E_2 - 18k\ln C = 2.7 \times 10^{-21}$ J/K.

(d) The definition of the temperature is given by $\frac{1}{T} \equiv \frac{\partial S}{\partial E}$. At the equilibrium, it is found from (b) that $\frac{\partial \ln\Omega_1}{\partial E_1} = \frac{\partial \ln\Omega_2}{\partial E_2}$, or $\frac{1}{T_1} \equiv \frac{\partial k\ln\Omega_1}{\partial E_1} = \frac{\partial k\ln\Omega_2}{\partial E_2} \equiv \frac{1}{T_2}$. Therefore, $T_1 = T_2 = 1/\frac{\partial(k\ln\Omega_1)}{\partial E_1} = 1/\frac{\partial(k\ln\Omega_2)}{\partial E_2} = \frac{E_1}{10k} = \frac{E_2}{8k} = 4.0 \times 10^3$ K.