

Physics 425, Spring 2009

Quiz - 6
April 21, Tuesday

Name: _____ ID: _____

(books and notes are not allowed)

Planck constant: $h = 6.63 \times 10^{-34} \text{ J s}$

Boltzmann constant: $k = 1.38 \times 10^{-23} \text{ J/K}$

Avogadro number: $N_a = 6.02 \times 10^{23}$

Gas constant: $R = 8.31 \text{ J/K/mol}$

$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$

$T (\text{K}) = T (^{\circ} \text{C}) + 273.16$

$\log(e) = 0.43$

$$\text{occupation number : } n_s = \frac{1}{e^{(\epsilon_s - \mu)/kT} \pm 1}$$

Note:

- Not all the above constants or relations are useful.
- Explain how the result is obtained. Generous partial credit will be given to correct steps and explanations even if the final result may be erroneous.

1. Suppose that a gas consists of N ($N \gg 1$) identical particles (i.e., it is not a photon gas).
 - (a) The occupation number n_s is the mean number of particles occupying a certain quantum state s . If the gas particles are Bosons, can you give the range of values that n_s can take? I.e., if $n_0 < n_s < n_1$, what should be n_0 and n_1 ?
 - (b) For Bosons, in what situations (e.g., at what temperature and quantum state), $n_s = n_0$? And in what situations, $n_s = n_1$? Please explain why.
 - (c) If the gas particles are Fermions, what should be n_0 and n_1 ?
 - (d) For Fermions, at the temperature $T = 0$, if the chemical potential of the gas is μ , find the occupation number for the state with energy ε . [Hint: distinguish the $\varepsilon > \mu$ and $\varepsilon < \mu$ cases.]
 - (e) We learned that the kinetic energy of the translational motion of classical particles is described by equipartition $E = 3/2 NkT$. At sufficiently low temperatures, we expect that the total kinetic energy would be different from this expression, because particles can no longer be considered classical particles. Suppose that at low temperatures, $E = 3/2 NkT + \delta$, and δ is the correction term. Should $\delta > 0$ or $\delta < 0$, for Fermions and Bosons, respectively? Please explain how you reach your conclusion. [Hint: consider the shape of n_s as a function of energy for three kinds of particles.]

Key to solutions:

- (a) Bosons can occupy a single quantum state by any number. Therefore, $0 < n_s < N$.
- (b) At very low temperature (T approaches zero), bosons will condense into the ground state, so at the ground state, $n_s = N$, and at excited states, $n_s = 0$. On the other hand, when temperature is very high or particle density is very low, the number of quantum states is a lot larger than the number of particles, then at any state, $n_s \ll 1$.
- (c) No more than 1 Fermion can occupy a certain state, so $0 < n_s < 1$.
- (d) Here we shall look at the behavior of n_s when T approaches zero. From the expression $n_s = \frac{1}{e^{(\varepsilon_s - \mu)/kT} + 1}$, it is seen that when $T = 0$: for states with energy $\varepsilon > \mu$, the exponent becomes $+\infty$, so that $n_s \approx e^{-\infty} = 0$; for states with energy $\varepsilon < \mu$, the exponent becomes $-\infty$, and the exponential term becomes zero, so that $n_s = \frac{1}{0+1} = 1$. This result means that, at very low temperatures, particles will fill quantum states one by one up to the energy state of $\varepsilon = \mu$, and no particles occupy those states with higher energies. This result also

means that the chemical potential of Fermions at $T = 0$ is determined by the particle number N .

- (e) We may consider this problem in the following ways: the curves of the occupation number for the three kinds of particles are such that Bosons have the sharpest distribution with a larger fraction of particles occupying the low energy states in comparison with classical particles, while Fermions exhibit a flatter distribution with a larger fraction of particles in high energy states. Therefore, the total energy of Bosons should be lower, or $\delta < 0$, while the total energy of Fermions should be larger, or $\delta > 0$. Further, we may think about the physical picture of particles occupying quantum states at low temperatures. All Bosons will tend to occupy the ground state (the lowest energy state), while Fermions have to fill the states with increasing energies one by one. The case for the classical particles will be in between. As a result, the total energy of Bosons is lower, and the total energy for Fermions is larger.

