

III.5 Quantum Statistics

Class schedule: April 14, 16

Reading assignments: S19, R9.1-9.3

Examples marked with * are important and should be studied carefully.

Review: **classical statistics** and applications: Boltzman factor, mean values, equipartition, specific heats, and paramagnetism.

The **quantum statistics** examine the number of particles, the **occupation number**, for a particular quantum state, while the classical statistics look at the probability of a single particle in a certain state. Suppose that in thermal equilibrium, there are n_s particles at the energy state ϵ_s , the probability of such a situation is $P_s = C e^{-n_s(\epsilon_s - \mu)/kT}$, where μ is the chemical potential of the gas particles. The mean number of particles in this state, or the occupation number, is given by

$$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-n_s(\epsilon_s - \mu)/kT}}{\sum_{n_s} e^{-n_s(\epsilon_s - \mu)/kT}}$$

The sum is performed over all possible values of n_s .

Through some mathematical operations, it is seen that the occupation number can be obtained by $\bar{n}_s = -kT \partial \ln(Z_s) / \partial \epsilon_s$, where $Z_s = \sum_{n_s} e^{-n_s(\epsilon_s - \mu)/kT}$.

To perform the sum, we need to know what values n_s is allowed to take, based on which, different quantum distributions are given:

Fermi-Dirac distribution: no more than one particle can occupy a certain quantum state, i.e., $n_s = 0, 1$ only. These particles are called **Fermions**. Particles with half integer spins, like electrons and protons, are Fermions.

Bose-Einstein distribution: any number of particles can occupy a given state, i.e., $n_s = 0, 1, 2, \dots$. These particles are called **Bosons**. Particles with integer number of spins, like photons, are Bosons.

Example 1: the Pauli exclusion principle and the ground state at low temperature.

The FD and BE distributions can be calculated

$$\bar{n}_s = \frac{1}{e^{(\epsilon_s - \mu)/kT} \pm 1}$$

with + referring to FD distribution and - referring to BE distribution.

At the classical limit, both the BE and FD distributions become the classical Maxwell-Boltzmann distribution

$$\bar{n}_s = N \frac{e^{-\epsilon_s/kT}}{\sum_s e^{-\epsilon_s/kT}}$$

Example 2: the probability of two particles found at the same state of a three-state system in the case of MB, FD, or BE distribution.

*Example 3: properties of MB, FD, and BE distributions.

The classical limit is met when

$$\left(\frac{V}{N}\right)^{1/3} \gg \frac{h}{\sqrt{3mkT}}$$

, in which case, the occupation number of any state is much smaller than 1, so there is no distinction between FD and BE distributions any more, both becoming identical to the classical MB distribution.

Example 4: the classical limits for neutral gas at the room temperature and ionized gas (e.g. protons and electrons) in stellar atmosphere and interior.

The total number, energy, and mean values of particles may be calculated from the occupation number of all accessible states by $N = \sum_s \bar{n}_s$, $E = \sum_s \bar{n}_s \epsilon_s$, and $\bar{y} = \sum_s \bar{n}_s y_s$. If accessible states can be described by a continuous spectrum (i.e., the gap between any two states is very small) $g(\epsilon)d\epsilon$, then we obtain the total number and energy by $N = \int g(\epsilon)\bar{n}(\epsilon)d\epsilon$, $E = \int g(\epsilon)\bar{n}(\epsilon)\epsilon d\epsilon$, and $\bar{y} = \frac{1}{N} \int g(\epsilon)\bar{n}(\epsilon)y(\epsilon)d\epsilon$.