

Derivation

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Motivation: Roger Scott developed a circular model illustrating how a magnetic intrusion in the supra-arcade current sheet affects the plasma flow. The next step in this research is to transform the simple circular intrusion into an elliptical intrusion complicating the flow pattern. To transform a circular intrusion to an elliptical intrusion, we began with the magnetic flux function representative of a circle of unit radius.

Let us declare B as magnetic field, when $B \rightarrow$ constant at $r \gg R$ and $B \cdot \hat{r} = 0$ on a circle. We know that when $\nabla \cdot B = \nabla \times B = 0 \Rightarrow B \propto \nabla f$. If $z \in \mathbf{C}$, $z = x + iy$ and x represents the real axis.

Since z is an element of the complex number set, we must go back to this definition using real numbers. So let us redefine f as,

$$f = x \left(1 - \frac{1}{x^2 + y^2} \right) \quad (8)$$

Thus, the flux function is given by,

$$f = x \left(1 - \frac{1}{r^2} \right) \quad \text{such that} \quad \nabla^2 f = 0 \quad (1)$$

$$\text{and } B = (\hat{x} \times \hat{y}) \times \nabla f \quad (2)$$

Now we will require useful facts that simplify the problem:

At the line $y = 0$

Provided that $\zeta(z)$ is analytic then, $\nabla^2 f(\zeta(z)) = 0$.

$$\nabla g = \partial_x g \quad (9)$$

The traditional formula for a Joukowski Transformation,

$$J \circ z = Az + B \frac{1}{z} \quad (3)$$

is a conformal mapping therefore, under the conditions (1-3) displayed above, the inverse Joukowski Transform,

$$\zeta = J' \circ z = \frac{1}{2A} \left(z \pm \sqrt{z^2 - 4AB} \right). \quad (4)$$

is an analytic function. So it follows that g to represent $f(\zeta(z))$, then

$$\nabla^2 g(z) = 0 \quad (5)$$

This also allows us to say that

$$g = f(\zeta) \quad (6)$$

We are ultimately interested in the magnetic field strength,

$$|B| = |\nabla g| = |\nabla f(\zeta(z))| \quad (7)$$

This implies that on this line

$$\text{Im}(z) = \text{Im}(\zeta) = 0 \quad (10)$$

This allows us to assume

$$f(x) = x \left(1 - \frac{1}{x^2} \right) \quad (11)$$

and that

$$(J' \circ z) = (J' \circ x) = \frac{1}{2A} \left(x \pm \sqrt{x^2 - 4AB} \right). \quad (12)$$

This all leads us to the conclusion that,

$$|B| = \partial_x [f(J' \circ x)] \quad (13)$$

The next goal is to derive $\partial_x g(x)$.

Let us begin by allowing ζ to represent $J' \circ x$, thus,

$$\partial_x f = \partial_\zeta \cdot f \cdot \frac{d\zeta}{dx} \quad (14)$$

After considering the chain rule, we are left with,

$$\partial_\zeta f(\zeta) \cdot \partial_x(\zeta(x)) \quad (15)$$

A simple substitution creates the equation,

$$\partial_\zeta f(\zeta) = \partial_\zeta \left(\zeta \left(1 - \frac{1}{\zeta^2} \right) \right) \quad (16)$$

Using the product rule we can rewrite this as,

$$\left(1 - \frac{1}{\zeta^2} \right) + \zeta \left(\frac{2}{\zeta^3} \right) \quad (17)$$

This is equal to,

$$\left(1 - \frac{1}{\zeta^2} \right) + \left(\frac{2}{\zeta^2} \right), \quad (18)$$

which reduces to,

$$1 + \frac{1}{\zeta^2} \quad (19)$$

Now we will work on the derivation of the second part. The first step is substituting the solution for equation(12) in for ϖ .

$$\partial_x(\zeta(x)) = \partial_x \left(\frac{x}{2A} \pm \left(\frac{\sqrt{x^2 - 4AB}}{2A} \right) \right). \quad (20)$$

After pulling out a $\frac{1}{2A}$, and taking the derivative the right hand side simplifies to,

$$\frac{1}{2A} \left(1 \pm \left(\frac{2x}{2\sqrt{x^2 - 4AB}} \right) \right). \quad (21)$$

Finally, the complete equation is,

$$\partial_x g(\zeta(x)) = \left(1 + \frac{1}{\zeta^2} \right) \left(1 \pm \left(\frac{x}{\sqrt{x^2 - 4AB}} \right) \right). \quad (22)$$

Right now we have an equation in two variables but we know that $x = A\zeta + 1/\zeta$, so lets isolate the equation into terms of *zeta*.