

# Dimensionless Quantities, Logarithmic Axes, and Thinking Like a Physicist

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December 2, 2010

## Abstract

Two time-honored tricks allow physicists to reduce equations to their most essential elements and represent them graphically. Understanding *dimensionless quantities* and *logarithmic axes* will empower you to make sense of the universe.

planets orbit. By dividing all orbital periods by the Earth's period, and all orbital semimajor axes by that of Earth,<sup>1</sup> we have created *dimensionless* (unitless) measures of time and space. These particular dimensionless quantities are relevant wherever the gravitational potential is provided by a one solar mass object.

## 1 Dimensionless Quantities

For any physically significant parameter, there is always at least one physically meaningful *scale*. For example, an appropriate length scale for Earth is one Earth radius. Changes of position that are comparable to this value will typically place you in a very different regime: from land to ocean, from the surface to outer space, or into the deep interior of the planet. It is often helpful to report measurements in units of the relevant physical scale.

### 1.1 Kepler's Third Law

Within our solar system, for example, it makes sense to measure distances in AU, and time in years. In these units, Kepler's third law does not require any constant of proportionality:

$$T_{yr}^2 = a_{AU}^3. \quad (1)$$

The above equation is *prima facie* absurd, because it seems to say that the units of years<sup>2</sup> and AU<sup>3</sup> are the same. Well, in a sense they are. This is because years and AU are related by the characteristics of the central potential (solar gravity) in which all the

### 1.2 AC Circuits

Let us apply this concept to simple AC circuits made from passive components (inductors, resistors and capacitors). We know that such "linear" circuits respond at the frequency of the driving voltage. Moreover, the impedances of the components all depend on the angular frequency  $\omega$ . As a result, interesting physical quantities like the total impedance of the circuit or its transfer function will also be functions of frequency. It is usually fruitful to identify a characteristic value of the frequency,  $\omega_c$ .<sup>2</sup> We can work in units of the characteristic frequency by defining a *dimensionless frequency*,

$$\xi = \frac{\omega}{\omega_c}.$$

Given the units of the three passive components,

Component	Units
$R$	$V/A = \Omega$
$L$	$H = Vs/A = \Omega s$
$C$	$F = C/V = As/V = s/\Omega$

<sup>1</sup>1AU is not exactly the semimajor axis of Earth's orbit, but it is very close.

<sup>2</sup>This is equivalent to identifying a characteristic time,  $T_c = 2\pi/\omega$ .

each of the three two-component combinations has a characteristic frequency, and a corresponding dimensionless frequency variable:

Combination	$\omega_c$	$\xi$
$RC$	$1/RC$	$\omega RC$
$LR$	$R/L$	$\omega L/R$
$LC$	$1/\sqrt{LC}$	$\omega\sqrt{LC}$

In class, we analyzed the  $RC$  highpass filter. I could have written the magnitude of the gain any number of ways, for example

$$G = \frac{R}{\sqrt{\frac{1}{\omega^2 C^2} + R^2}} = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}.$$

The second version is better because it makes it easy to see that  $\omega RC$  is unitless: This “dimensionless frequency” turns out to be the only variable of interest in the equation. I also express the solution to the highpass filter problem as a *gain*, which is the ratio  $V_{out}/V_{in}$ . In effect, this is a dimensionless version of  $V_{out}$ . In dimensionless units then, every  $RC$  high-pass filter, regardless of the input voltage amplitude or the values of  $R$  and  $C$ , is described by the simple relation

$$G = \frac{\xi}{\sqrt{1 + \xi^2}}, \quad (2)$$

where  $\xi \equiv \omega RC$ .

## 2 Logarithmic Axes

When plotting some quantity  $x$  that varies over many orders of magnitude, a linear scale tends to compress all the smaller values of  $x$  into a tiny fraction of the plot. This can be avoided by using logarithmic axes. Any sort of graphing software (even a cheesy spreadsheet) will give you the option of selecting logarithmic scales for the horizontal axis, vertical axis, or both. The idea is simply that we graph  $\log_{10}(x)$  instead of just  $x$ . As a result, there is equal spacing between  $x = 0.01, 0.1, 1, 10, \dots$ , because the logarithms of these values are equally spaced:  $\log_{10}(x) = -2, -1, 0, 1, \dots$ . It is traditional to label the logarithmic axis with the values of  $x$  rather than the values of its logarithm.

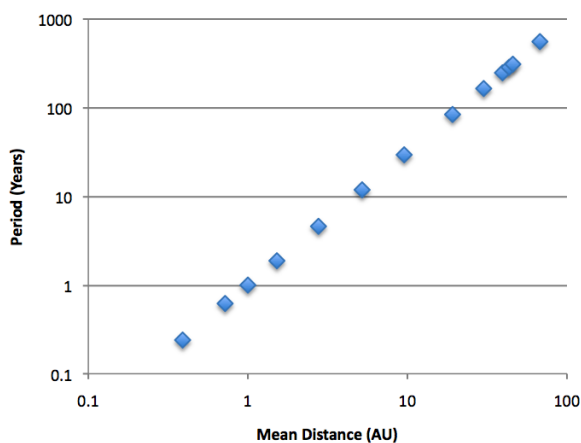


Figure 1: Orbital parameters of the planets and dwarf planets, plotted on log-log axes, using units normalized to Earth values.

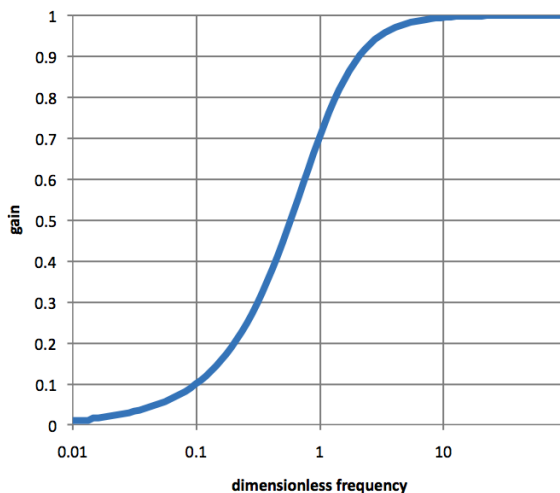


Figure 2: Relationship between frequency and gain for an  $RC$  high pass filter (equation 2). The log frequency axis conveniently spans four orders of magnitude.

Figure 1 illustrates Kepler’s third law (equation 1). The planets are almost uniformly distributed in log

coordinates, whereas in linear coordinates the inner solar system would be collapsed almost to a single point. Moreover, by using logarithmic scales for both axes, the power law relationship between period and distance is viewed as a straight line (more on that in §2.1).

In figure 2, the dimensionless frequency  $\xi$  spans four orders of magnitude. The logarithmic frequency axis makes it possible to see in detail how the gain asymptotically approaches zero at low frequencies. On a linear scale, everything left of  $\xi = 1$  would look like a vertical line at the left border of the plot. Since equation 2 is all in terms of dimensionless quantities, one graph suffices to describe the behavior of *all possible RC highpass filters*.

## 2.1 Quick Sketches

What if the final exam presents you with the dreaded “sketch the behavior as a function of frequency”? Often, the simplicity of a dimensionless expression makes it possible to make a rough plot without resorting to a computer or even a calculator. If we take the logarithm of both sides of equation 1, it turns out that Kepler’s Third Law is just a line with a slope of  $3/2$  in log-log coordinates:

$$\log T = \frac{3}{2} \log a.$$

The relationship between  $T$  and  $a$  is called a “power law” with an “index” of  $3/2$ .

The highpass filter (equation 2) is also straightforward, if you think in terms of *limiting cases*. For small frequencies ( $\omega RC \ll 1$ ), the response approaches zero asymptotically. For large frequencies, ( $\omega RC \gg 1$ ), the gain approaches unity. There is a transition between these two asymptotic behaviors around the critical value  $\omega = \omega_c = 1/RC$ , where  $G = 1/\sqrt{2}$ . So, by looking just at just three cases (small frequencies, large frequencies, and the critical frequency), you can learn enough to sketch  $G$  vs.  $\omega RC$  (compare figure 2).

## 3 Further Reading

Our discussion of dimensional analysis has been entirely *ad hoc*. There is, however, a mathematically rigorous understanding of the usefulness of dimensionless quantities. See the Wikipedia articles on dimensionless quantities ([http://en.wikipedia.org/wiki/Dimensionless\\_number#List\\_of\\_dimensionless\\_quantities](http://en.wikipedia.org/wiki/Dimensionless_number#List_of_dimensionless_quantities)) and the Buckingham  $\pi$  theorem.

Some of the most fruitful applications of dimensionless numbers are in fluid mechanics. The Reynolds number, for example, is a dimensionless quantity that classifies several different regimes of fluid flow ranging from simple laminar behavior to periodic shedding of vortices to turbulence. Have a look at the Wikipedia article on the Reynolds number. An image search for Reynolds Number also turns up some interesting results.