

SO FAR:

$$V = IR$$

KCL, KVL

SERIES & PARALLEL

NODE VOLTAGE ANALYSIS

THEVENIN & NORTON EQUIVALENTS

BJTs, DIODES

OP AMPS

AMPLIFIERS & ANALOG COMPUTERS

COMPARATORS & SCHMITT TRIGGERS

TODAY:

CH. 51, COMPLEX NUMBERS (USED THROUGHOUT PHYSICS!)

TODAY'S LECTURE PREPARES US TO DEAL WITH CAPACITORS AND INDUCTORS, USING COMPLEX IMPEDANCE.

COMING UP:

ALTERNATING CURRENT (CH. 7)

CAPACITORS, INDUCTORS, IMPEDANCE (CH. 8)

AC CIRCUIT ANALYSIS (CH. 9-10)

TRANSIENT RESPONSE (CH. 12-13)

COMPLEX NUMBERS (CH. 51)

$$j \equiv \sqrt{-1}$$

EXAMPLE:

$$x^2 + 1 = 0$$

SOLVE FOR x :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{0 - 4}}{2} = \pm \sqrt{-1} = \pm j$$

EXAMPLE:

$$2x^2 - 2x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot 1}}{4} = \frac{+2 \pm 2\sqrt{-1}}{4} = \frac{1}{2} \pm \frac{1}{2}j$$

MATHEMATICIANS AND PHYSICISTS NORMALLY USE $i = \sqrt{-1}$, WHILE EE'S USE $j = \sqrt{-1}$. SINCE OUR CURRENTS ARE i , WE WILL FOLLOW THE EE CONVENTION.

REAL NUMBERS, \mathbb{R} — WHAT DOES $x \in \mathbb{R}$ MEAN?

IMAGINARY NUMBERS, \mathbb{I}

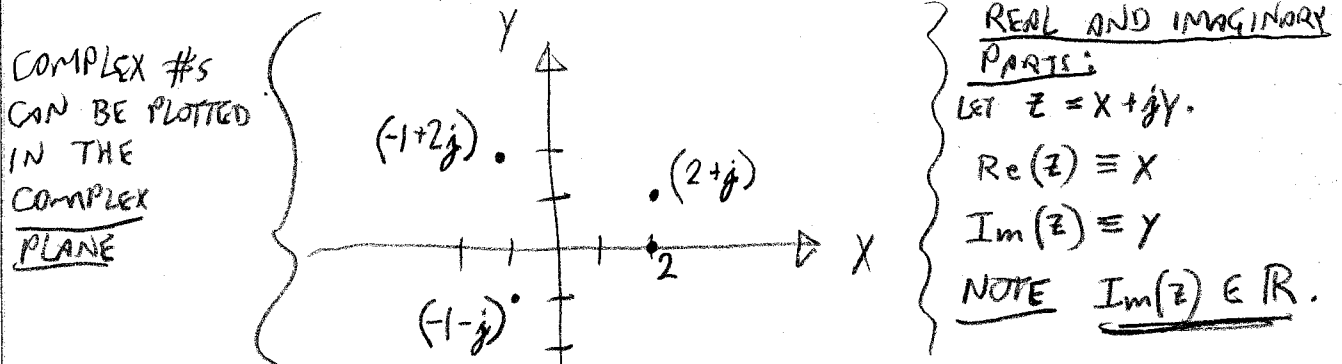
CAN BE EXPRESSED AS $z = jy$, $y \in \mathbb{R}$

THE IMAGINARY NUMBERS ARE NOT CLOSED UNDER MULTIPLICATION:

$$z^2 = (jy)^2 = -y^2 \in \mathbb{R}$$

COMPLEX NUMBERS, \mathbb{C}

CAN BE EXPRESSED $z = x + jy$; $x, y \in \mathbb{R}$



★ ALL QUADRATICS HAVE 2 COMPLEX ROOTS.

COMPLEX CONJUGATE ($*$):

LET $Z = x + jy$. THE COMPLEX CONJUGATE OF Z IS DEFINED:

$$Z^* \equiv x - jy$$

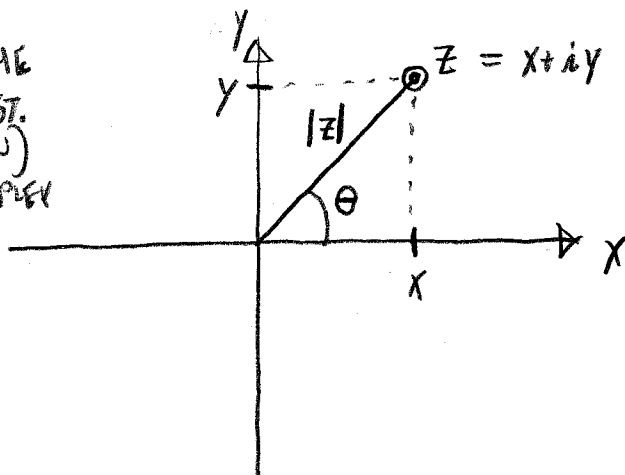
THE ABSOLUTE SQUARE OF Z IS DEFINED:

$$|Z|^2 = Z^*Z = (x - jy)(x + jy) = x^2 + y^2$$

THE MAGNITUDE OF Z IS DEFINED:

$$|Z| = \sqrt{|Z|^2} = \sqrt{x^2 + y^2}$$

$|Z|$ IS THE RADIUS (DIST. FROM ORIGIN) IN THE COMPLEX PLANE.



THE "PHASE ANGLE" θ IS RELATED TO x & y :

$$\tan \theta = \frac{y}{x}$$

LET $a \in \mathbb{R}$. IT IS EASY TO SHOW THAT:

$$|aZ| = a|Z|$$

TO SIMPLIFY A RATIO OF COMPLEX NUMBERS, MULTIPLY THE DENOMINATOR BY ITS OWN COMPLEX CONJUGATE:

$$\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + j \frac{bc - ad}{c^2 + d^2}$$

CANPAD

THE EXPONENTIAL FUNCTION CAN BE DEFINED BY ITS POWER SERIES:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

WHAT HAPPENS WHEN WE GIVE IT AN IMAGINARY ARGUMENT, $iy \in \mathbb{I}$ ($y \in \mathbb{R}$) ?

$$\begin{aligned} e^{iy} &= 1 + iy - \frac{y^2}{2!} - i \frac{y^3}{3!} + \frac{y^4}{4!} + i \frac{y^5}{5!} \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots \right) + i \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots \right) \end{aligned}$$

$$e^{iy} = \cos y + i \sin y$$

THIS IS EULER'S FORMULA * $e^{iy} \in \mathbb{C}$.

y IS IN RADIANS.

EXAMPLE

WRITE THE REAL AND IMAGINARY PARTS OF $\exp(x + iy)$.

$$\begin{aligned} e^{x+iy} &= e^x e^{iy} \\ &= e^x (\cos y + i \sin y) \end{aligned}$$

$$\text{Re}(e^{x+iy}) = e^x \cos y$$

$$\text{Im}(e^{x+iy}) = e^x \sin y$$

THE MAGNITUDE OF e^{iy} IS UNITY:

$$|e^{iy}| = \cos^2 y + \sin^2 y = 1$$

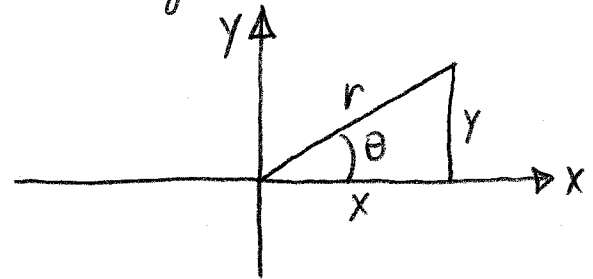
$$\text{ALSO, } |e^{x+iy}| = e^x |e^{iy}| = e^x.$$

* SOMETIMES INCORRECTLY CALLED EULER'S THEOREM, WHICH IS DIFFERENT.

AMPAI

SUPPOSE WE WANT TO REPRESENT A COMPLEX NUMBER IN POLAR COORDINATES:

$$Z = x + jy$$



$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\text{SO } Z = r \cos \theta + j r \sin \theta$$

$$= r (\cos \theta + j \sin \theta)$$

$$Z = r e^{j\theta}$$

INTERPRETATION: $r = |z|$, EUCLIDEAN DISTANCE FROM THE ORIGIN
 $e^{j\theta}$ = UNIT VECTOR AT ANGLE θ .

WE CAN CONVERT FROM POLAR TO RECTANGULAR COORDINATES USING

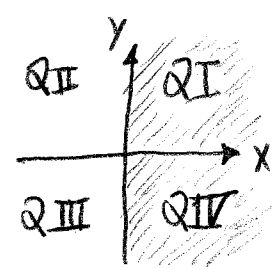
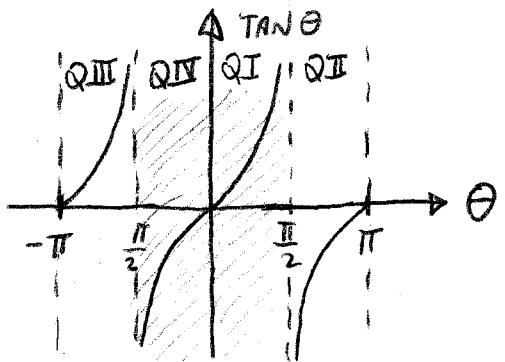
$$\theta = \text{TAN}^{-1}\left(\frac{y}{x}\right),$$

BUT THERE'S A 180° (π RADIANS) AMBIGUITY:

$$\frac{-y}{-x} = \frac{y}{x}, \quad \text{SO } \text{TAN}^{-1}\left(\frac{-y}{-x}\right) = \text{TAN}^{-1}\left(\frac{y}{x}\right)$$

$$\text{I.E. } \boxed{\text{TAN } \theta = \text{TAN}(\theta \pm \pi)}$$

SIMPLE ATAN()
 FUNCTIONS GIVE
 ANSWERS IN
 THE SHADED
 REGION, QI/QIV,
 $-\pi/2 < \theta \leq \pi/2$



WORK AROUND THE AMBIGUITY BY NOTICING WHICH QUADRANT $x + jy$ IS IN.

SUPPOSE WE WISH TO SIMPLIFY THE EXPRESSION:

$$z = \frac{\sqrt{2}e^{j\pi/4}}{-1+\sqrt{3}j}$$

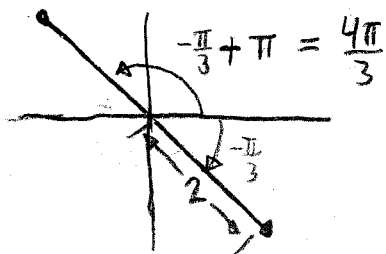
I. IN POLAR COORDINATES:

$$-1+\sqrt{3}j = re^{j\theta}, \quad r = \sqrt{3+1} = 2$$

$$\theta = \tan^{-1}(-\sqrt{3}) = -1.047 = -\frac{\pi}{3} \quad (-60^\circ)$$

Q: IS θ CORRECT?

A: NO:



SINCE $x < 0$ AND $y > 0$,
WE WANT QII RATHER
THAN QIII.

SO $\theta = \frac{4\pi}{3}$.

$$z = \frac{\sqrt{2}e^{j\pi/4}}{2e^{j4\pi/3}} = \frac{1}{\sqrt{2}}e^{-j(\frac{\pi}{4}-\frac{4\pi}{3})} = \frac{1}{\sqrt{2}}e^{-j13\pi/12}$$

II. IN RECTANGULAR COORDINATES:

$$\sqrt{2}e^{j\pi/4} = \sqrt{2}\cos\frac{\pi}{4} + j\sqrt{2}\sin\frac{\pi}{4} = 1 + j$$

$$z = \frac{(1+j)(-1-\sqrt{3}j)}{(-1+\sqrt{3}j)(-1-\sqrt{3}j)} = \frac{(-1+\sqrt{3}) - (1+\sqrt{3})j}{4}$$

$$z = \frac{-1+\sqrt{3}}{4} - \frac{1+\sqrt{3}}{4}j$$

TREAT j AS YOU WOULD ANY OTHER ALGEBRAIC SYMBOL.
IT OBEYS ALL THE USUAL RULES.