

Gauss's Theorem

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1 Introduction

The fundamental theorem of calculus may seem obvious to you by now:

$$\int_a^b \left(\frac{d}{dx} f(x) \right) dx = f(b) - f(a). \quad (1)$$

What might surprise you is the range of interesting forms this theorem takes when we generalize it to three dimensions. Griffiths §1.3 discusses the gradient theorem, Gauss's theorem, and Stokes' theorem but provides justification only the gradient theorem. I will derive Gauss's theorem in the next section.

2 Deriving Gauss's Theorem

We wish to prove the divergence theorem, also called Gauss's theorem, which states:

$$\boxed{\int_V \nabla \cdot \mathbf{v} \, d\tau = \oint_{\partial V} \mathbf{v} \cdot \mathbf{d}\mathbf{a}} \quad (2)$$

where \mathbf{v} is a vector function of position, and ∂V is the closed surface bounding volume V . The vector surface element is $\mathbf{d}\mathbf{a} = \hat{\mathbf{n}} \, da$, where the surface normal vector $\hat{\mathbf{n}}$ is directed outward.

Gauss's theorem is a scalar equation, but nevertheless the integrals on both sides of the equation are each a sum of three parts, corresponding to x, y, z components of the vector \mathbf{v} . Making an equation of just the v_x parts,

$$\int_V \frac{\partial v_x}{\partial x} \, d\tau = \oint_{\partial V} v_x \hat{\mathbf{x}} \cdot \mathbf{d}\mathbf{a}. \quad (3)$$

If we can show that equation 3 is true, then all three components of equation 2 must similarly be true, and Gauss's theorem is proved.

The volume V is represented by the brown "potato" in figure 1. Imagine dividing the potato into a large number of "french fries", each of them parallel to

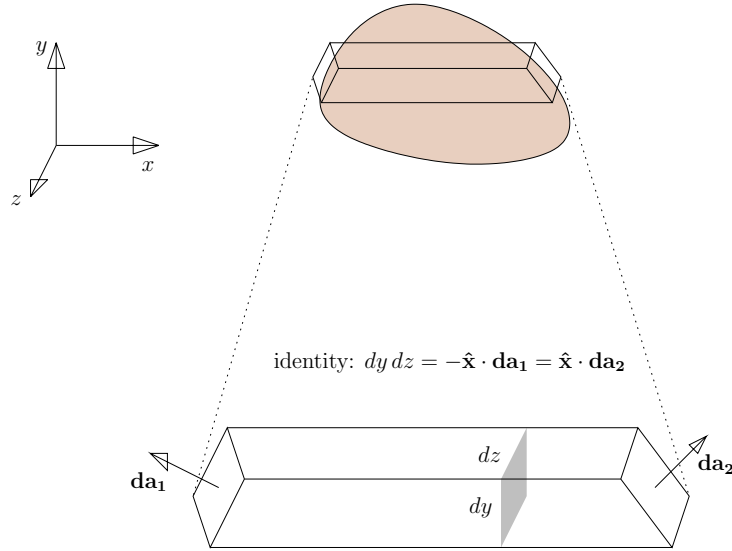


Figure 1: Building up the volume and surface integrals over the “potato” from a collection of “french fries”.

the x -axis. One such french fry is illustrated in the figure. The volume integral on the left side of equation 3 is given by integrating over all the french fries:

$$\int_V \frac{\partial v_x}{\partial x} d\tau = \int dy dz \int_{x_1}^{x_2} \frac{\partial v_x}{\partial x} dx \quad (4)$$

Using the fundamental theorem of calculus, this becomes

$$= \int dy dz [v_x(x_2) - v_x(x_1)] = \int_{\partial V_2} v_x(x_2) dy dz - \int_{\partial V_1} v_x(x_1) dy dz, \quad (5)$$

where ∂V_1 and ∂V_2 are the left and right surfaces of the “peel” enclosing the potato. Of course v_x is really a function of all three coordinates; I am just trying to make the notation compact. Taking advantage of the geometrical identity in figure 1, the integral can be rewritten

$$= \int_{\partial V_2} v_x(x_2) \hat{\mathbf{x}} \cdot \mathbf{da}_2 + \int_{\partial V_1} v_x(x_1) \hat{\mathbf{x}} \cdot \mathbf{da}_1 = \oint_{\partial V} v_x \hat{\mathbf{x}} \cdot \mathbf{da} \quad \checkmark \quad (6)$$

We have proven equation 3, which in turn proves Gauss’s theorem (equation 2). Notice that our proof assumed a very simple potato, in the sense that that each french fry would intersect the peel just twice. A potato with a more complicated shape can be constructed from multiple simple potatoes, so no generality is lost by this assumption. You might want to quickly sketch that out for yourself; for example, if we join two banana-shaped potatoes to make a u-shaped potato, what happens at the shared surface where they meet?