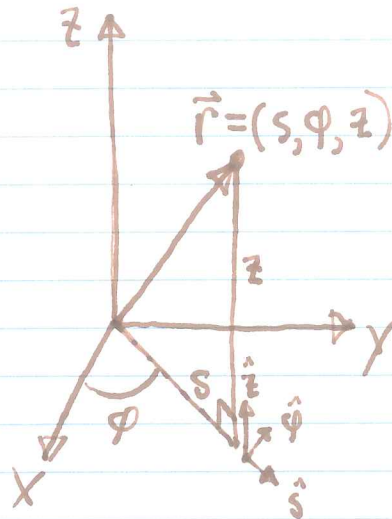


# CURVILINEAR COORDINATES (GRIFFITHS §1.4)

## CYLINDRICAL COORDINATES:



$\vec{r} = (s, \phi, z) \neq s\hat{s} + \phi\hat{\phi} + z\hat{z} !$  DISCUSS

Q: WHAT IS  $\vec{r}$ ?  
 A:  $s\hat{s} + z\hat{z} = \vec{r}$ .

IN OTHER WORDS,  
 $\phi \neq r_{\phi} = 0$ .

$\hat{s} \times \hat{\phi} = \hat{z}$  (RIGHT HANDED)

NOT CONSTANT

$$\begin{cases} \hat{s} = \hat{x} \cos \phi + \hat{y} \sin \phi \\ \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi \\ \hat{z} = \hat{z} \end{cases}$$

$$\begin{cases} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{cases} \Rightarrow s = \sqrt{x^2 + y^2}$$

$d\vec{l} = \hat{s} ds + \hat{\phi} s d\phi + \hat{z} dz$

$dV = s ds d\phi dz$

$\nabla T = \frac{\partial T}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial T}{\partial \phi} \hat{\phi} + \frac{\partial T}{\partial z} \hat{z}$ .

$\nabla \cdot \vec{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_{\phi}}{\partial \phi} + \frac{\partial v_z}{\partial z}$ .

$\nabla \times \vec{v} = \left( \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_{\phi}}{\partial z} \right) \hat{s} + \left( \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right) \hat{\phi} + \frac{1}{s} \left( \frac{\partial}{\partial s} (s v_{\phi}) - \frac{\partial v_s}{\partial \phi} \right) \hat{z}$ .

POSITION

Q: HOW WOULD YOU SUBTRACT TWO ~~DISPLACEMENT~~ VECTORS?  
 E.G.  $\vec{r} = \vec{r} - \vec{r}'$ .

A: CONVERT TO CARTESIAN AND SUBTRACT!

EXAMPLE

LAST TIME, WE INVESTIGATED A FIELD OF RIGID ROTATION:

$$\vec{v} = r \omega \hat{\phi} \quad (\text{VELOCITY OF A RIGIDLY ROTATING BODY})$$

IN FACT, ~~STAND~~ IT WOULD BE MORE APPROPRIATE TO WRITE

$$\vec{v} = s \omega \hat{\phi}.$$

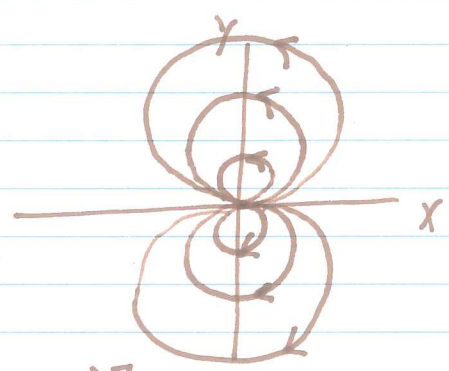
WE CONVERTED TO CARTESIAN COORDINATES TO CALCULATE THE CURL. BUT THIS IS EASIER:

$$\begin{aligned} \nabla \times (s \omega \hat{\phi}) &= -\frac{\partial v_{\phi}}{\partial r} \hat{s} + \frac{\hat{z}}{s} \frac{\partial (s v_{\phi})}{\partial s} \\ &= 2\omega \hat{z} \checkmark \end{aligned}$$

EXERCISE

FOR  $\vec{v} = \frac{1}{s^2} (\hat{s} \cos \phi + \hat{\phi} \sin \phi)$ , SKETCH THE FIELD & CALCULATE THE DIVERGENCE & CURL.

$$\begin{aligned} \nabla \cdot \vec{v} &= \frac{1}{s} \frac{\partial}{\partial s} \left( \frac{\cos \phi}{s} \right) + \frac{1}{s} \frac{\partial}{\partial \phi} \left( \frac{\sin \phi}{s^2} \right) \\ &= \frac{1}{s} \left( -\frac{\cos \phi}{s} \right) + \frac{1}{s} \left( \frac{\cos \phi}{s} \right) = 0. \end{aligned}$$



$$\begin{aligned} \nabla \times \vec{v} &= 0 \hat{s} + 0 \hat{\phi} + \frac{1}{s} \left[ \frac{\partial}{\partial s} \left( \frac{\sin \phi}{s} \right) - \frac{\partial}{\partial \phi} \left( \frac{\cos \phi}{s^2} \right) \right] \hat{z} \\ &= \frac{1}{s} \left[ \frac{-\sin \phi}{s^2} + \frac{\sin \phi}{s^2} \right] \hat{z} = 0. \end{aligned}$$

EXAMPLE LET  $\vec{v} = \vec{r}$ . CHECK THE DIVERGENCE THEOREM ON THE CYLINDER  $s = R, 0 < z < h$ .

$\vec{v} = \vec{r} = s\hat{s} + z\hat{z}; \nabla \cdot \vec{v} = \frac{1}{s} \frac{\partial}{\partial s}(s^2) + \frac{\partial z}{\partial z} = 2 + 1 = 3.$

$\int (\nabla \cdot \vec{v}) dV = \int_0^h \int_0^{2\pi} \int_0^R 3 s ds d\phi dz = 3\pi R^2 h$  FAMILIAR RESULT.

WALLS OF CYLINDER:  $\vec{v} = R\hat{s} + z\hat{z}; d\vec{a} = \hat{s} R d\phi dz$   
 $\vec{v} \cdot d\vec{a} = R^2 d\phi dz$

$\therefore \int_W \vec{v} \cdot d\vec{a} = R^2 \int_0^h \int_0^{2\pi} d\phi dz = 2\pi R^2 h.$

BOTTOM OF CYLINDER:  $\vec{v} = s\hat{s} + 0\hat{z}; d\vec{a} = -\hat{z} s ds d\phi$   
 $\vec{v} \cdot d\vec{a} = 0$

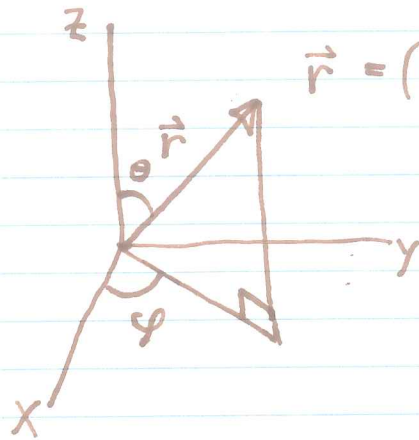
$\therefore \int_B \vec{v} \cdot d\vec{a} = 0.$

TOP OF CYLINDER:  $\vec{v} = s\hat{s} + h\hat{z}; d\vec{a} = \hat{z} s ds d\phi$   
 $\vec{v} \cdot d\vec{a} = h s ds d\phi$

$\therefore \int_T \vec{v} \cdot d\vec{a} = \int_0^{2\pi} \int_0^R h s ds d\phi = \pi R^2 h.$

SU  $\oint \vec{v} \cdot d\vec{a} = 2\pi R^2 h + 0 + \pi R^2 h = 3\pi R^2 h.$  ✓

## SPHERICAL COORDINATES:



$$\vec{r} = (r, \theta, \varphi) = r \hat{r}.$$

$$\hat{r} \times \hat{\theta} = \hat{\varphi}$$

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

$$\hat{r} = \sin \theta \cos \varphi \hat{x} + \sin \theta \sin \varphi \hat{y} + \cos \theta \hat{z}$$

$$\hat{\theta} = \cos \theta \cos \varphi \hat{x} + \cos \theta \sin \varphi \hat{y} - \sin \theta \hat{z}$$

$$\hat{\varphi} = -\sin \varphi \hat{x} + \cos \varphi \hat{y}$$

ALL VARY  
WITH ANGULAR  
VARIABLES.

$$d\vec{r} = \hat{r} dr + r d\theta \hat{\theta} + r \sin \theta d\varphi \hat{\varphi}.$$

$$dV = r^2 \sin \theta dr d\theta d\varphi$$

$$\nabla T = \frac{\partial T}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \varphi} \hat{\varphi}$$

$$\nabla \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\varphi}{\partial \varphi}$$

$$\nabla \times \vec{v} = \frac{\hat{r}}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\varphi) - \frac{\partial v_\theta}{\partial \varphi} \right] + \frac{\hat{\theta}}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \varphi} - \frac{\partial}{\partial r} (r v_\varphi) \right] + \frac{\hat{\varphi}}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right].$$

$$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \varphi^2}$$

EXAMPLE IN HW 01, IN CARTESIAN COORDINATES YOU CALCULATED  $\nabla \times \left( \hat{z} \times \frac{\hat{r}}{r} \right) = 2z \hat{r} / r^3$ .

(a) ★ VERIFY IN SPHERICAL COORDINATES, ★

NOTE THAT  $\hat{z} \times \hat{r} = \hat{\phi} \cos \theta - \hat{\psi} \sin \theta$ .

$$\nabla \times \left( \hat{\phi} \frac{\cos \theta}{r} \right) = \frac{\hat{r}}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \frac{\sin^2 \theta \cos \theta}{r} \right) \right] - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} (\cos \theta)$$

$$= \frac{\hat{r}}{r^2 \sin \theta} \left[ \cos^2 \theta - \sin^2 \theta \right]$$

$$= \frac{\hat{r}}{r^2 \sin \theta} (2 \sin \theta \cos \theta) = \frac{2 \cos \theta \hat{r}}{r^2} \quad \star$$

BUT  $z = r \cos \theta$ ; MULT. TOP & BOTTOM BY  $r$ ;

$$\nabla \times \left( \hat{z} \times \frac{\hat{r}}{r} \right) = \frac{2z \hat{r}}{r^3} \quad \checkmark$$

(b) WE EXPECT  $\nabla \cdot (\nabla \times \vec{v}) = 0$ . CHECK:

WE MAY EASILY CHECK THE DIVERGENCE FROM ★:

$$\nabla \cdot \left( \frac{2 \cos \theta \hat{r}}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (2 \cos \theta r) = 0. \quad \underline{\text{AS EXPECTED}}$$

(c) CHECK STOKES' THEOREM USING PATH  $r=R, \theta = \frac{\pi}{2}, 0 \leq \phi < 2\pi$ :

$$\oint \vec{v} \cdot d\vec{h} = ? \quad \vec{v} = \hat{z} \times \frac{\hat{r}}{r} = \frac{\hat{\phi} \sin \theta}{r} = \frac{\hat{\phi}}{r}, \quad d\vec{h} = \hat{\phi} r d\phi$$

$$\oint \vec{v} \cdot d\vec{h} = \int_0^{2\pi} d\phi = 2\pi$$

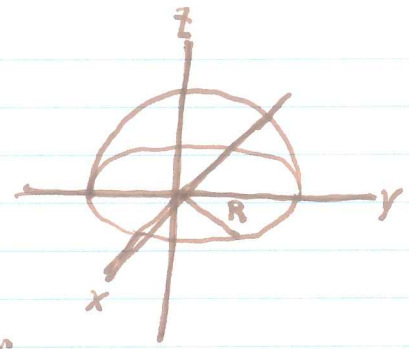
$$z=0 \text{ PLANE: } \int (\nabla \times \vec{v}) \cdot d\vec{a} = \int \left( \frac{2z \hat{r}}{r^3} \right) \cdot \hat{z} da = 0 \text{ SINCE } \hat{r} \cdot \hat{z} = 0 \text{ FOR } z=0? \quad \underline{\text{DISCUSS}}$$

★ EXERCISE: CHECK USING THE HEMISPHERE  $r=R, 0 \leq \theta \leq \pi/2$ .

EXERCISE, CONT.

ON THE HEMISPHERE,

$$\nabla \times \vec{v} = \frac{2 \cos \theta \hat{r}}{R^2}$$



$$d\vec{a} = R^2 \sin \theta \, d\theta \, d\phi \, \hat{r}$$

$$(\nabla \times \vec{v}) \cdot d\vec{a} = 2 \cos \theta \sin \theta \, d\theta \, d\phi$$

$$\int (\nabla \times \vec{v}) \cdot d\vec{a} = 2 \int_0^{2\pi} \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \, d\phi$$

$$= 4\pi \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta$$

$$= 4\pi \left[ \frac{1}{2} \sin^2 \theta \right]_0^{\pi/2} = 2\pi \checkmark$$