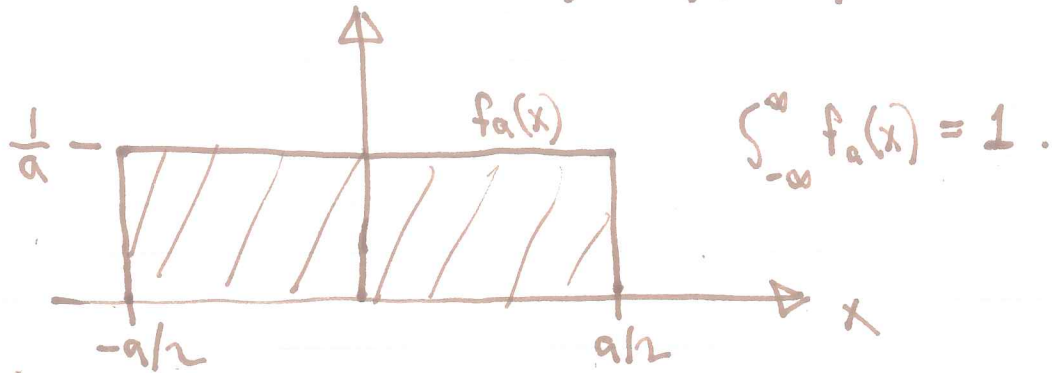


①

## DIRAC'S DELTA FUNCTION, §1.5

CONSIDER  $f_a(x) = \begin{cases} \frac{1}{a}, & |x| \leq a/2 \\ 0, & |x| > a/2 \end{cases} \quad (a > 0)$



DEFINE  $\delta(x) \equiv \lim_{a \rightarrow 0} f_a(x).$

THIS IS THE DIRAC DELTA FUNCTION. (DIRAC, 1930)

THERE ARE MULTIPLE ALTERNATIVE DEFINITIONS USING GAUSSIANS, ETC. BUT THE ESSENTIAL PROPERTIES ARE:

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

AND  $\int_{-\infty}^{\infty} \delta(x) dx = 1.$

CONSEQUENTLY,

$$\boxed{\int_{-\infty}^{\infty} f(x) \delta(x-b) dx = f(b).} \quad \underline{\text{DISCUSS.}}$$

Q:  $\int_{-\infty}^{\infty} f(x) \delta(b-x) dx = ?$

A:  $f(b).$   $\delta$  IS AN EVEN FUNCTION!

EXAMPLES

(a)  $I = \int_{-\infty}^{\infty} \sin(x) \delta(x + \frac{\pi}{2}) dx = ?$   
 $x + \frac{\pi}{2} = 0$  WHEN  $x = -\frac{\pi}{2}$ .  
 $\therefore I = \sin(-\frac{\pi}{2}) = -1$

(b)  $\int_0^{\infty} e^x \delta(x-a) dx = \begin{cases} e^a, & a > 0 \\ 0, & a < 0 \end{cases}$

(c)  $I = \int_0^{\infty} f(x) \delta(x^2-2) dx$  ← HOW ABOUT  $f(\sqrt{2})$ ? DISCUSS.  
 LET  $x = \sqrt{y}$ ,  $dx = \frac{1}{2} y^{-1/2} dy$   
 $I = \int_0^{\infty} f(\sqrt{y}) \delta(y-2) \frac{1}{2} y^{-1/2} dy$   
 $= f(\sqrt{2}) \frac{1}{2} 2^{-1/2} = \frac{f(\sqrt{2})}{2\sqrt{2}}$

(d)  $I = \int_{-\infty}^{\infty} f(x) \delta(c[x-b]) dx$   
 LET  $y = cx$ ,  $dx = \frac{1}{c} dy$ . IF  $c > 0$ ,  
 $I = \int_{-\infty}^{\infty} f(\frac{y}{c}) \delta(y-cb) \frac{1}{c} dy = \frac{1}{c} f(b)$ .

BUT SINCE  $\delta$  IS AN EVEN FUNCTION, THE SIGN OF  $c$  DOES NOT MATTER.

$$\int_{-\infty}^{\infty} f(x) \delta([x-b]c) dx = \frac{f(b)}{|c|}$$

THIS IS A USEFUL IDENTITY.

MULTI-DIMENSIONAL  $\delta$ -FUNCTIONS

CONSIDER  ~~$\delta^3(\vec{r}) \equiv \delta(x)\delta(y)\delta(z)$~~   $\delta^3(\vec{r}) \equiv \delta(x)\delta(y)\delta(z)$ .

EXERCISE

WHAT'S  $\int \delta^3(\vec{r}) dV$  ?

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z) dx dy dz \\ &= \int_{-\infty}^{\infty} \delta(x) dx \int_{-\infty}^{\infty} \delta(y) dy \int_{-\infty}^{\infty} \delta(z) dz \\ &= 1 \cdot 1 \cdot 1 = 1. \end{aligned}$$

AS YOU MIGHT SUPPOSE,

$$\delta^3(\vec{r}) \delta^3(\vec{r}-\vec{a}) = \begin{cases} 0 & \text{FOR } \vec{r} \neq \vec{a}, \\ \infty & \text{FOR } \vec{r} = \vec{a} \end{cases}$$

AND MOST IMPORTANTLY,

$$\int_V f(\vec{r}) \delta^3(\vec{r}-\vec{a}) dV = \begin{cases} f(\vec{a}), & \vec{a} \in V, \\ 0, & \vec{a} \notin V. \end{cases}$$

IS IT GOOD FOR ANYTHING?

EXAMPLE

VERIFY THE DIVERGENCE THEOREM FOR  $\vec{v} = \hat{r}/r^2$  ON THE SPHERE  $r=R$ .

EXERCISE

VERIFY THE STOKES' THEOREM FOR  $\vec{v} = \hat{\phi}/s$  ON THE CIRCLE  $s=R$  ( $z$  ARBITRARY).