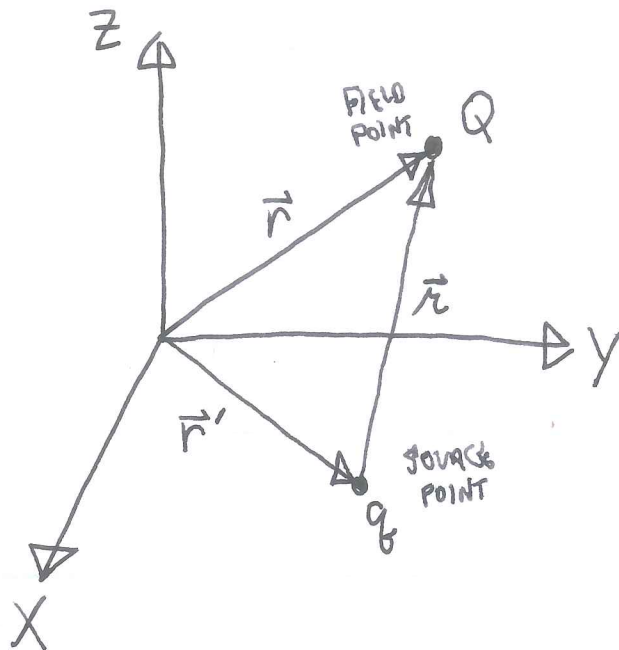


GRIFFITHS CH. 2 — ELECTROSTATICS



FORCE ON Q:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{r}$$

(COULOMB'S LAW)

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{q}{r^2} \hat{r}$$

SUPERPOSITION:

SUPPOSE N CHARGES,  $q_1, q_2, \dots, q_N$  ARE ALL INTERACTING WITH Q.

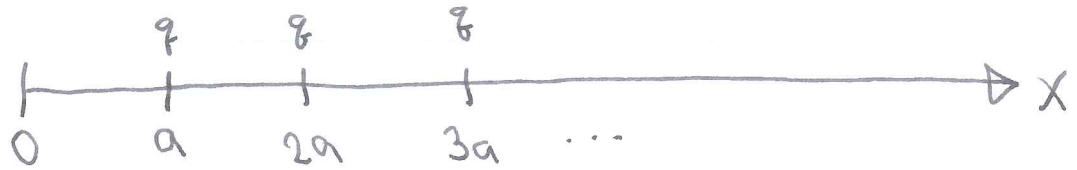
$$\vec{F} = \frac{Q}{4\pi\epsilon_0} \sum_{i=1}^N \frac{q_i \hat{r}_i}{r_i^2}$$

$$\vec{E} = \vec{F}/Q$$

NOTE THAT  $\vec{E} = \vec{E}(\vec{r})$ . IT'S EVERYWHERE IN SPACE.

EXAMPLE

CONSIDER AN INFINITE SEQUENCE OF CHARGES ON THE X-AXIS:



FIND  $\vec{E}$  AT  $\vec{r} = 0$ .

$$\vec{E}_0 = \frac{-\hat{x} q}{4\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{n^2 a^2} = \frac{-\hat{x} q}{4\pi\epsilon_0 a^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

THE SUM IS THE FAMOUS "BASEL PROBLEM" SOLVED BY EULER:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\therefore \vec{E}_0 = \frac{-\pi q \hat{x}}{24 a^2 \epsilon_0}$$

★ HOW WOULD YOU SET UP THE PROBLEM FOR A GENERAL FIELD POINT  $\vec{r}$ ?

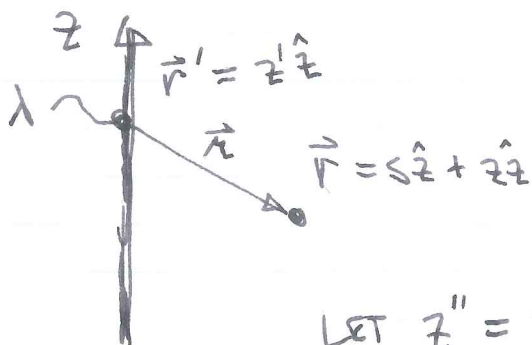
CONTINUOUS CHARGE DISTRIBUTIONS:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{r}}{r^2} dq$$

$$dq = \begin{cases} \rho dV' & \dots \text{VOLUME CHARGE DENSITY, } \rho, \text{ COUL/M}^3 \\ \sigma da' & \dots \text{SURFACE CHARGE DENSITY, } \sigma, \text{ COUL/M}^2 \\ \lambda dl' & \dots \text{LINEAR CHARGE DENSITY, } \lambda, \text{ COUL/M} \end{cases}$$

WHY ARE THE INTEGRATION VARIABLES PRIMED? - DISCUSS.

EXAMPLE FIELD DUE TO A CONSTANT LINE CHARGE,  $\lambda$ , INFINITE IN EXTENT.



$$\vec{r} = \vec{r} - \vec{r}' = s\hat{s} + (z - z')\hat{z}$$

$$\vec{E} = \int_{-\infty}^{\infty} \frac{\lambda (s\hat{s} + (z - z')\hat{z}) dz'}{4\pi\epsilon_0 (s^2 + (z - z')^2)^{3/2}}$$

LET  $z'' = z - z'$ ; ELIMINATE  $s\hat{s}$  BY SYMMETRY.

THEN WE HAVE:

$$\vec{E} = \frac{\lambda \hat{s}}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz''}{(s^2 + z''^2)^{3/2}}$$

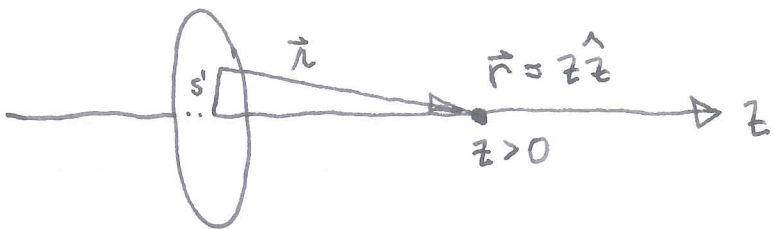
WE CAN SEE BY INSPECTION THAT THE  $\int \propto \frac{1}{s^2}$ .

$$\text{SO } \vec{E} \propto \frac{\lambda \hat{s}}{\epsilon_0 s^2}$$

CRC # 165:  $\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}}$  (EASY TO VERIFY)

$$\text{SO } \vec{E} = \frac{\lambda \hat{s}}{2\pi\epsilon_0 s^2} = \frac{\lambda \hat{s}}{2\pi\epsilon_0 s}$$

EXAMPLE: UNIFORMLY CHARGED DISK



$$\begin{aligned} \vec{r}' &= s' \hat{s}' \\ \vec{r} &= z \hat{z} \\ \vec{R} &= z \hat{z} - s' \hat{s}' \\ R &= [z^2 + s'^2]^{1/2} \end{aligned}$$

$$\begin{aligned} \vec{E}(z) &= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma \vec{R} da}{R^2} = \frac{\sigma}{4\pi\epsilon_0} \int_{s'=0}^R \int_{\phi=0}^{2\pi} \frac{(z \hat{z} - s' \hat{s}') s' d\phi' ds'}{(z^2 + s'^2)^{3/2}} \quad \text{(SYMMETRY)} \\ &= \frac{\sigma}{2\epsilon_0} \int_{s'=0}^R \frac{z \hat{z} s' ds'}{(z^2 + s'^2)^{3/2}} = \frac{\sigma}{2\epsilon_0} \frac{-z \hat{z}}{(z^2 + s'^2)^{1/2}} \Big|_{s'=0}^R \\ &= -\frac{\sigma z \hat{z}}{2\epsilon_0} \left[ \frac{1}{\sqrt{z^2 + R^2}} - \frac{1}{z} \right] = \frac{\sigma z \hat{z}}{2\epsilon_0} \left[ 1 - \frac{z}{\sqrt{z^2 + R^2}} \right] \end{aligned}$$

IF WE HOLD FIXED THE TOTAL CHARGE  $Q = \pi R^2 \sigma$ ,  
AND TAKE THE LIMIT  $R \rightarrow 0$ ,

$$\vec{E} = \frac{Q \hat{z}}{2\pi \epsilon_0} \left[ \frac{1}{R^2} - \frac{z}{R^2 \sqrt{z^2 + R^2}} \right]$$

BINOMIAL EXPANSION  
 $(1 + \epsilon)^x = 1 + x\epsilon + O(\epsilon^2)$

$$\left[ \right] = \left[ \frac{\sqrt{z^2 + R^2} - z}{R^2 \sqrt{z^2 + R^2}} \right] \approx \left[ \frac{z \left( 1 + \frac{R^2}{2z^2} \right) - z}{R^2 \sqrt{z^2 + R^2}} \right]$$

$$= \left[ \frac{1}{2z \sqrt{z^2 + R^2}} \right] \rightarrow \frac{1}{2z^2}$$

$$\therefore \vec{E} \rightarrow \frac{Q \hat{z}}{4\pi \epsilon_0 z^2}$$

✓ WE RECOVER COULOMB'S LAW FOR A POINT CHARGE.

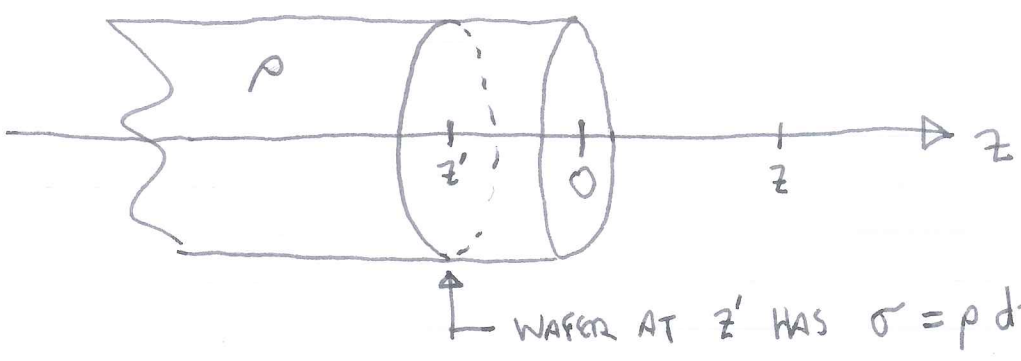
ANOTHER INTERESTING LIMIT: HOLD  $\sigma$  FIXED,  $R \rightarrow \infty$  OR  $z \rightarrow 0$ :

$$\vec{E} = \frac{\sigma \hat{z}}{2\epsilon_0}$$

— INFINITE SHEET OF CHARGE (NO DEPENDENCE ON  $z$ !)

WHAT IF  $z < 0$  ?

WE MAY EXTEND THE PREVIOUS EXAMPLE TO CONSIDER A SEMI-INFINITE ROD WITH CHARGE DENSITY  $\rho = \text{CONST.}$



FIELD AT  $z$  DUE TO WAFER AT  $z'$ :

$$d\vec{E} = \frac{\rho \hat{z}}{2\epsilon_0} \left[ 1 - \frac{z-z'}{\sqrt{(z-z')^2 + R^2}} \right] dz' \quad \left. \vphantom{\frac{\rho \hat{z}}{2\epsilon_0}} \right\} \text{USING FORMULA FOR DISK.}$$

FIELD FROM SEMI-INFINITE ROD:

$$\begin{aligned} \vec{E} &= \int d\vec{E} = \int_{-\infty}^0 \frac{\rho \hat{z}}{2\epsilon_0} \left[ 1 - \frac{z-z'}{\sqrt{(z-z')^2 + R^2}} \right] dz' \\ &= \frac{\rho \hat{z}}{2\epsilon_0} \left[ z' + \left[ (z-z')^2 + R^2 \right]^{1/2} \right]_{-\infty}^0 \\ &= \frac{\rho \hat{z}}{2\epsilon_0} \left\{ \sqrt{z^2 + R^2} - \lim_{x \rightarrow \infty} \left[ -x + \left[ (z+x)^2 + R^2 \right]^{1/2} \right] \right\} \end{aligned}$$

$$\vec{E} = \frac{\rho \hat{z}}{2\epsilon_0} \left\{ \sqrt{z^2 + R^2} - z \right\}$$

THE LIMIT GOES LIKE THIS: AS  $x \rightarrow \infty$ ,

$$\begin{aligned} -x + \left[ (z+x)^2 + R^2 \right]^{1/2} &\rightarrow \left( z^2 + 2xz + x^2 + R^2 \right)^{1/2} - x \\ &\rightarrow x \left( \frac{z^2}{x^2} + 2\frac{z}{x} + 1 + \frac{R^2}{x^2} \right)^{1/2} - x \rightarrow x \left( 1 + 2\frac{z}{x} \right)^{1/2} - x \\ &\rightarrow x \left( 1 + \frac{z}{x} \right) - x \rightarrow z. \quad \checkmark \end{aligned}$$

BINOMIAL EXPANSION  
 $(1+\epsilon)^\alpha = 1 + \alpha\epsilon + O(\epsilon^2)$