

## GRIFFITHS § 2.2, GAUSS'S LAW

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IN ELECTROSTATICS, OUR ONLY SOURCE OF ELECTRIC FIELD IS ELECTRIC CHARGE. GIVEN SOME CHARGE DENSITY  $\rho(\vec{r})$ , THE ELECTRIC FIELD IS GIVEN BY COULOMB'S LAW

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}') \hat{r}}{r^2} dV',$$

WHERE  $\hat{r} = \vec{r} - \vec{r}'$ .

NOW, WHAT IS  $\nabla \cdot \vec{E}$ ?

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{r}') \left[ \nabla \cdot \left( \frac{\hat{r}}{r^2} \right) \right] dV'$$

Q: HOW WAS I ABLE TO MOVE THE DIVERGENCE OPERATOR INTO THE INTEGRAL? WHY DOES IT HIT ONLY THE FACTORS THAT DEPEND ON  $\hat{r}$ ?

A: ONLY  $\hat{r}$  HAS  $\vec{r}$  IN IT. THE INTEGRAL AND ITS INDEPENDENT VARIABLES ARE IN THE PRIMED COORDINATE SYSTEM.

$$\begin{aligned} \nabla \cdot \left( \frac{\hat{r}}{r^2} \right) &= \nabla \cdot \left( \frac{\hat{r}}{r^3} \right) = \nabla \cdot \left\{ \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \right\} \\ &= \frac{3}{[\ ]^{3/2}} - \frac{3(x-x')^2}{[\ ]^{5/2}} - \frac{3(y-y')^2}{[\ ]^{5/2}} - \frac{3(z-z')^2}{[\ ]^{5/2}} = 0. \end{aligned}$$

FROM X-COMPONENT

BUT CONSIDER THE SPHERE  $r=R$ , CENTERED AT  $\vec{r} = \vec{r}'$ .

$$\oint \left( \frac{\hat{r}}{r^2} \right) \cdot d\vec{a} = \oint \left( \frac{\hat{r}}{r^2} \right) \cdot \hat{r} da = \frac{1}{R^2} \oint da = \frac{1}{R^2} 4\pi R^2 = 4\pi.$$

IMPLICATION: IF THE DIVERGENCE THEOREM IS TO BE SATISFIED, WE MUST CONCLUDE

$$\nabla \cdot \left( \frac{\hat{r}}{r^2} \right) = 4\pi \delta^3(\vec{r}) = \delta^3(\vec{r} - \vec{r}'). 4\pi$$

$$\therefore \nabla \cdot \vec{E} = \frac{4\pi}{4\pi\epsilon_0} \int \rho(\vec{r}') \delta(\vec{r} - \vec{r}') dV'$$

$$\nabla \cdot \vec{E} = \rho(\vec{r}) / \epsilon_0$$

GAUSS'S LAW  
(DIFFERENTIAL FORM)

Q: IS THERE ANY NEW  
PHYSICS IN THIS?

DISCUSS.

A: COULOMB'S FORCE LAW + THE  
IDEA OF A "FIELD" + MATH  
= GAUSS'S LAW.

EXAMPLE: SUPPOSE  $\vec{E} = \overset{\text{CONSTANT}}{E_0} \hat{s}$ . FIND  $\rho(\vec{r})$ .

$$\nabla \cdot \vec{E} = \frac{1}{s} \frac{\partial}{\partial s} (s E_s) = \frac{E_0}{s} = \rho / \epsilon_0.$$

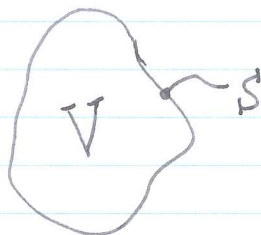
$$\therefore \rho = \frac{\epsilon_0 E_0}{s}.$$

EXAMPLE:  $\vec{E} = \overset{\text{CONSTANT}}{A} \hat{s} / s$ . FIND  $\rho(\vec{r})$

$$\nabla \cdot \vec{E} = \frac{1}{s} \frac{\partial}{\partial s} (s E_s) = 0. \quad \rho = 0? \quad \text{DISCUSS!}$$

GAUSS'S LAW IS  $\nabla \cdot \vec{E} = \rho / \epsilon_0$ . APPLY THE DIVERGENCE THEOREM:

$$\int_V (\nabla \cdot \vec{E}) dV = \oint_S \vec{E} \cdot d\vec{a}$$



$$\int_V (\rho / \epsilon_0) dV = \oint_S \vec{E} \cdot d\vec{a}$$

$$\therefore \oint_S \vec{E} \cdot d\vec{a} = Q_{\text{ENC}} / \epsilon_0$$

GAUSS'S LAW  
(INTEGRAL FORM)

EXAMPLE: LET  $\rho(\vec{r}) = f(r)$ . FIND  $\vec{E}(\vec{r})$ .

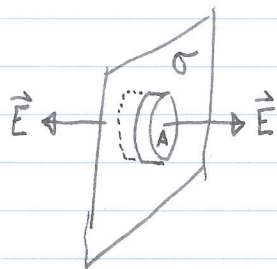
BY SYMMETRY, WE SUPPOSE THAT  $\vec{E}(\vec{r}) = E(r) \hat{r}$ .

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int \rho dV \quad \text{--- ON } \underline{r=R}.$$

$$\Rightarrow 4\pi R^2 E(R) = \frac{1}{\epsilon_0} \int_0^R 4\pi r^2 f(r) dr$$

$$E(R) = \frac{1}{\epsilon_0 R^2} \int_0^R f(r) r^2 dr.$$

EXAMPLE: UNIFORM SURFACE CHARGE  $\sigma$  ON A PLANE:



$$\left. \begin{aligned} \oint \vec{E} \cdot d\vec{a} &= 2EA \\ \frac{1}{\epsilon_0} \int \rho dV &= \sigma A / \epsilon_0 \end{aligned} \right\} E = \sigma / 2\epsilon_0$$

COMPARE RESULT FROM LAST TIME  
VIA COULOMB'S LAW!