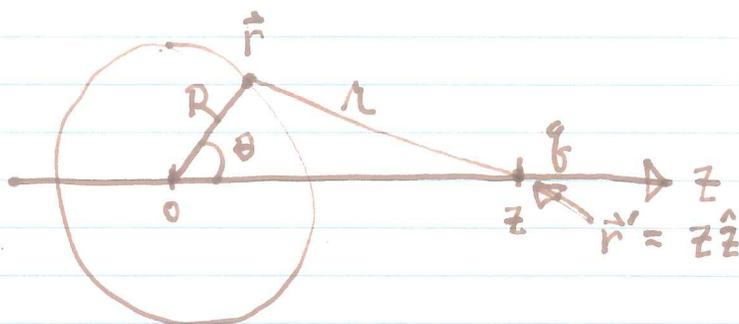


GRIFFITHS §3.1

→ WHAT IS THE CHARACTER OF SOLUTIONS TO $\nabla^2 V = 0$ ON \mathbb{R}^3 ?

LET'S IMAGINE A SPHERE OF RADIUS R , WITH NO CHARGE DENSITY ($\rho = 0$) INSIDE. IT'S JUST EMPTY SPACE, IN WHICH $\nabla^2 V = 0$.



NOW INTRODUCE A CHARGE q ON THE z -AXIS, $z > R$.

Q: WHAT IS $V(\vec{r})$ AT A POINT ON THE SURFACE OF THE SPHERE?

A: $V = \frac{q}{4\pi\epsilon_0 r}$, $\vec{r} = \vec{r} - \vec{r}'$
 $= (R \cos\theta \hat{z} + R \sin\theta \hat{s}) - z \hat{z}$

$$r = \sqrt{(R \cos\theta - z)^2 + R^2 \sin^2\theta}$$

$$= \sqrt{R^2 + z^2 - 2Rz \cos\theta}$$

← WE COULD HAVE USED THE LAW OF COSINES!

$$\therefore V = \frac{q}{4\pi\epsilon_0 \sqrt{R^2 + z^2 - 2Rz \cos\theta}}$$

Q: WHAT IS ~~THE~~ THE MEAN VALUE OF V ON THE SURFACE OF THE SPHERE?

$$\begin{aligned}
A: \bar{V} &= \frac{1}{4\pi R^2} \oint V da \\
&= \frac{1}{4\pi R^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{q R^2 \sin\theta d\theta d\phi}{4\pi\epsilon_0 \sqrt{R^2+z^2-2Rz\cos\theta}} \\
&= \frac{q}{16\pi^2\epsilon_0} (2\pi) \int_{\theta=0}^{\pi} \frac{\sin\theta d\theta}{\sqrt{R^2+z^2-2Rz\cos\theta}} \\
&= \frac{q}{8\pi\epsilon_0} \sqrt{R^2+z^2-2Rz\cos\theta} \left(\frac{1}{zR} \right) \Big|_{\theta=0}^{\pi} \\
&= \frac{q}{8\pi\epsilon_0 zR} \left[\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz} \right]
\end{aligned}$$

SINCE $z > R$, THIS IS

$$\bar{V} = \frac{q}{8\pi\epsilon_0 zR} \left[(R+z) - (z-R) \right]$$

$$\therefore \boxed{\bar{V} = \frac{q}{4\pi\epsilon_0 z} = V(0)}$$

Q: HOW WOULD THIS CHANGE IF WE HAD A CHARGE DENSITY $\rho(r)$ (WITH $\rho=0$ FOR $r < R$)?

Q: HOW WOULD THIS RESULT CHANGE IF THE ORIGIN WAS MOVED ELSEWHERE?

CONCLUSION: THE CHARACTER OF SOLUTIONS TO $\nabla^2 V = 0$ IN 3D IS SUCH THAT

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_S V da,$$

WHEREAS S IS A SPHERE OF ANY RADIUS CENTERED AT \vec{r} .

COROLLARY: V CAN HAVE NO LOCAL MAXIMA NOR MINIMA. (DISCUSS!)

- \therefore
- THE EXTREME VALUES OF V OCCUR AT THE BOUNDARIES
 - IN FREE SPACE, WITH $V \rightarrow 0$ AS $r \rightarrow \infty$, $V = 0$ EVERYWHERE (WE KNEW THAT!)

IF TIME PERMITS: POINT RELAXATION DEMO