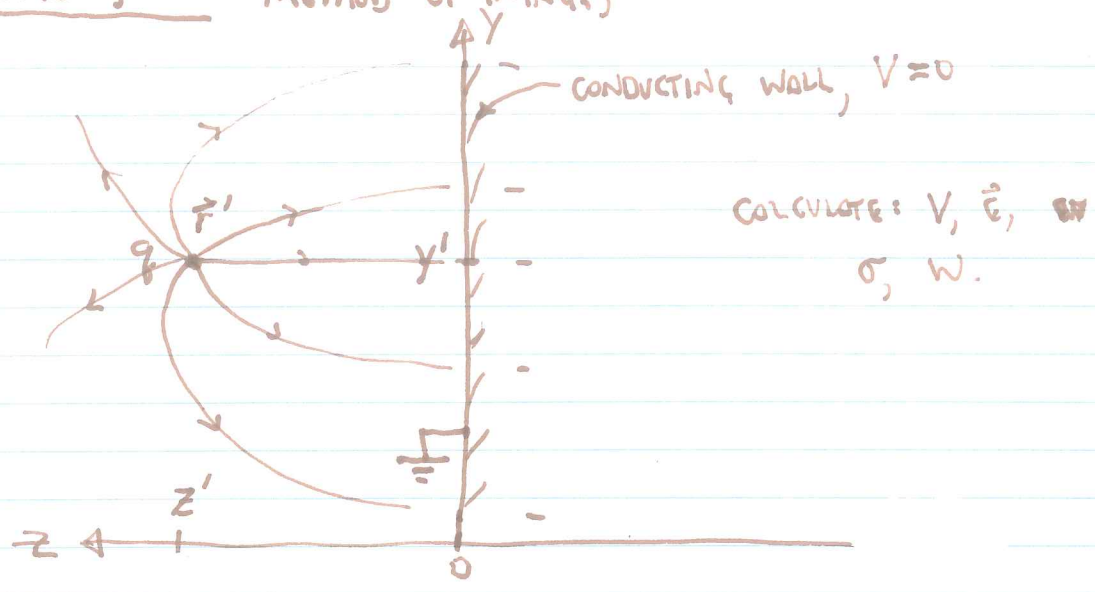


GRIFFITHS §3.2 METHOD OF IMAGES



THE POTENTIAL FROM q ~~IS~~ ONLY IS

$$V_q = \frac{q}{4\pi\epsilon_0 r} = \frac{q}{4\pi\epsilon_0 \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

WE HAVE A CONDUCTING WALL THAT IMPOSES A BOUNDARY CONDITION $V=0 @ z=0$. THE COMPLETE SET OF BOUNDARY CONDITIONS ALSO INCLUDES $V \rightarrow 0$ AS $r \rightarrow \infty$.

NOW, SUPPOSE WE HAD A CHARGE $-q$ AT $[x', y', -z']$. THE TOTAL POTENTIAL WOULD BE:

$$V = \frac{q}{4\pi\epsilon_0} \left[\left\{ (x-x')^2 + (y-y')^2 + (z-z')^2 \right\}^{1/2} - \left\{ (x-x')^2 + (y-y')^2 + (z+z')^2 \right\}^{1/2} \right]$$

Image $-q @ z = -z'$

NOTE THAT AT $z=0, V=0$. THIS SATISFIES THE B.C. IMPOSED BY THE WALL! ALSO $V \rightarrow 0$ AS $r \rightarrow \infty$.

BY THE FIRST UNIQUENESS THEOREM, THIS IS THE SOLUTION. NOTE THAT OUR SOLUTION IS VALID ONLY FOR $z \geq 0$. THE IMAGE CHARGE IS MERELY AN ARTIFICE THAT ALLOWS US TO CONSTRUCT A SOLUTION.

$$\vec{E} = -\nabla V; \text{ OR WRITE BY INSPECTION FROM THE 2 CHARGES.}$$

WE WOULD ALSO LIKE TO KNOW σ , w , # \vec{F}

$$\text{AT } \underline{z=0}, \vec{E} = \frac{q}{4\pi\epsilon_0} \left[\frac{(x-x')\hat{x} + (y-y')\hat{y} + (0-z')\hat{z}}{\{(x-x')^2 + (y-y')^2 + (0-z')^2\}^{3/2}} - \frac{(x-x')\hat{x} + (y-y')\hat{y} + (0+z')\hat{z}}{\{(x-x')^2 + (y-y')^2 + (0+z')^2\}^{3/2}} \right]$$

$$= \frac{q}{4\pi\epsilon_0} \frac{-2z'\hat{z}}{\{(x-x')^2 + (y-y')^2 + z'^2\}^{3/2}}$$

BY GAUSS'S LAW, $\sigma = \epsilon_0 E$ AT THE SURFACE.

$$\therefore \sigma = \frac{-qz'}{2\pi\{(x-x')^2 + (y-y')^2 + z'^2\}^{3/2}}$$

Q: IF WE WANT TO KNOW \vec{F} , IT WOULD MAKE SENSE TO PLUG IN $\vec{F} = q\vec{E}$, RIGHT?

A: ALMOST. LEAVE OUT THE FORCE OF q ON ITSELF!

$$\vec{F}_q = \frac{q^2(-\hat{z})}{4\pi\epsilon_0\{(x'-x)^2 + (y'-y)^2 + (z'+z)^2\}}$$

$$= \frac{-q^2\hat{z}}{16\pi\epsilon_0 z'^2} \quad \left(\begin{array}{l} \text{JUST THE FORCE ON} \\ q \text{ BY ITS IMAGE!} \end{array} \right)$$

Q: DOES THE FORCE GO IN $+\hat{z}$ IF WE SWITCH THE SIGN OF q ?

A: NO!

Q: WHAT IS THE FORCE ON THE ~~PLATE~~ ^{WALL}?

$$A: -\vec{F}_q = \frac{q^2\hat{z}}{16\pi\epsilon_0 z'^2}$$

Q: IS $W = \frac{1}{2} q V_{-q}$ (JUST q) OR $\frac{1}{2} q V_{-q} + \frac{1}{2} (q) V_q = \frac{1}{2} q V_{-q}$?

A: IT'S SAFER TO MOVE IN THE CHARGE q FROM INFINITY TO FIND THE WORK.

$$W = \frac{1}{4\pi\epsilon_0} \int_{\infty}^{z'} \frac{q^2}{4z^2} dz = \frac{-1}{4\pi\epsilon_0} \frac{q^2}{4d} = \frac{1}{2} q V_{-q}$$

Q: WHY NEGATIVE?

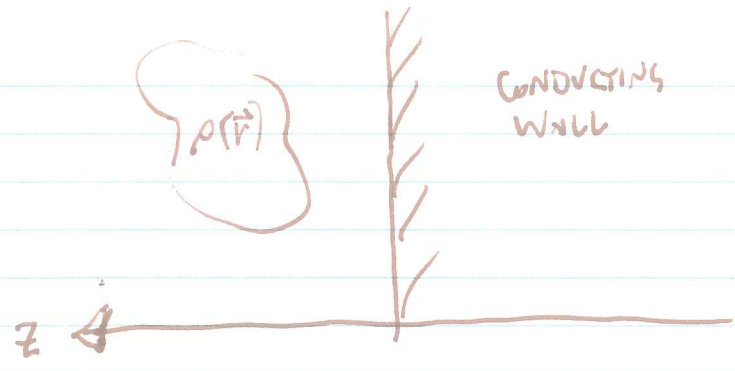
A: q AT r' HAS LOWER ENERGY THAN q AT $r \rightarrow \infty$

Q: WHY DOES ONLY THE REAL CHARGE COUNT, GIVING HALF THE ~~THE~~ WORK OF THE 2-CHARGE SYSTEM?

A: ONLY HALF THE VOLUME HAS THE E-FIELD OF THE 2-CHARGE SYSTEM, AND $W = \frac{1}{2} \epsilon_0 \int E^2 d\tau$.

(THIS IS JUST ONE OF MANY WAYS TO ANSWER THE QUESTION).

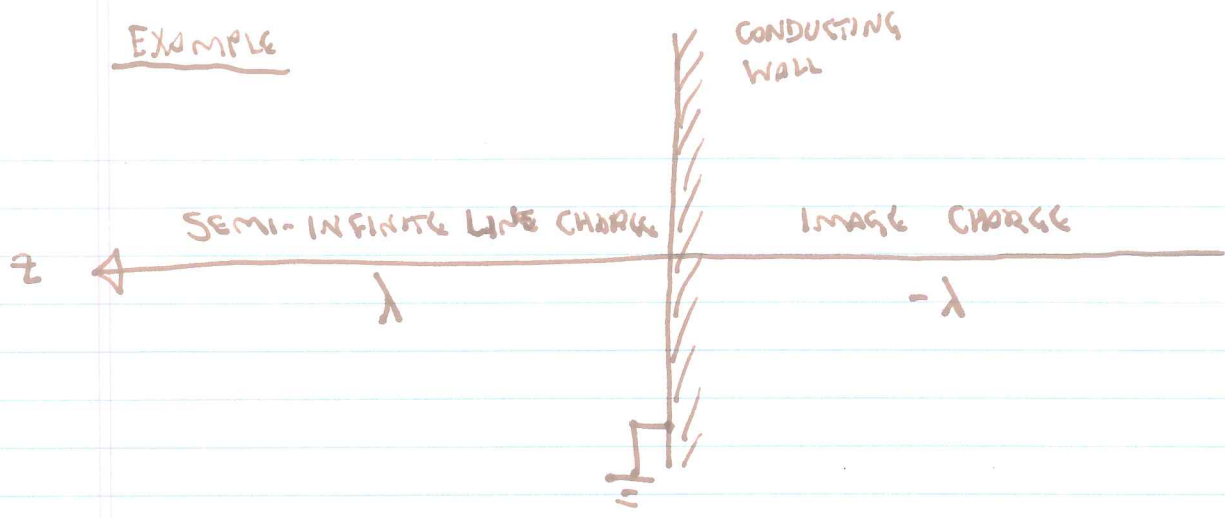
EXERCISE



WHAT IS V?

$$V = \int_{V'} \left[\frac{\rho(\vec{r}')}{4\pi\epsilon_0 r} + \frac{-\rho(\vec{r}' - \overset{\text{oops!}}{\cancel{2\hat{z}}})}{4\pi\epsilon_0 |\vec{r} - \vec{r}' + 2\hat{z}'z|} \right] dV'$$

EXAMPLE



RECALL OUR GENERAL SOLUTION WHEN V IS A FUNCTION OF θ ONLY:

$$V = C \ln\left(\tan \frac{\theta}{2}\right) + D$$

BOUNDARY: $0 = C \ln\left(\tan \frac{\pi}{4}\right) + D$
 $= 0 + D \Rightarrow \underline{D = 0}$

HOW DO WE SET C ? FOR SMALL θ (DISCUSS!),

$\vec{E} = E_s \hat{s}, E_s = -\frac{\partial V}{\partial r} = \frac{\lambda}{4\pi\epsilon_0 s}$
 FOR SMALL $\theta, s \approx r\theta, V \approx C \ln(\theta/2) \approx C(1 + \theta/2)$
 AND $ds = r d\theta$, SO

$$-\frac{dV}{ds} = -\frac{1}{r} \frac{dV}{d\theta} = \frac{-C}{r} \frac{1/2}{\theta/2}$$

SO WE HAVE $\frac{\lambda}{4\pi\epsilon_0 s} = \frac{-C}{\theta} = -\frac{C}{s}$

$$\therefore C = -\frac{\lambda}{4\pi\epsilon_0}$$

$$V = \frac{-\lambda}{4\pi\epsilon_0} \ln\left(\tan \frac{\theta}{2}\right)$$

Example

This is a simpler way to get the potential due to a line charge normal to a conducting plane. (Can you show that it is equivalent to the solution given on the handwritten notes?)

The diagram shows a vertical conducting plane on the right. A horizontal line charge with linear charge density λ_0 is located at a distance $2z$ to the left of the plane. A point P is located at a distance z to the right of the plane and a distance s vertically below the horizontal line charge. The electric field \vec{E} is shown pointing away from the line charge. The potential V is to be calculated at point P . The image charge is represented by a dashed line with charge density $-\lambda_0$ located at a distance z to the left of the plane.

WHAT IS $V(\vec{r})$?

$$V = \int \frac{dq}{4\pi\epsilon_0 r} = \int_{-\infty}^{\infty} \frac{\lambda(z') dz'}{4\pi\epsilon_0 \sqrt{s^2 + (z-z')^2}}$$

$$= \int_0^{2z} \frac{\lambda_0 dz'}{4\pi\epsilon_0 \sqrt{s^2 + (z-z')^2}} + \int_{2z}^{\infty} \frac{\lambda_0 dz'}{4\pi\epsilon_0 \sqrt{s^2 + (z-z')^2}} - \int_{-\infty}^0 \frac{\lambda_0 dz'}{4\pi\epsilon_0 \sqrt{s^2 + (z-z')^2}}$$

LET $\xi = z' - z$

$$V = \frac{\lambda_0}{4\pi\epsilon_0} \left\{ \int_{-z}^z \frac{d\xi}{\sqrt{s^2 + \xi^2}} + \int_z^{\infty} \frac{d\xi}{\sqrt{s^2 + \xi^2}} - \int_{-\infty}^{-z} \frac{d\xi}{\sqrt{s^2 + \xi^2}} \right\} = \frac{\lambda_0}{2\pi\epsilon_0} \int_0^z \frac{d\xi}{\sqrt{s^2 + \xi^2}}$$

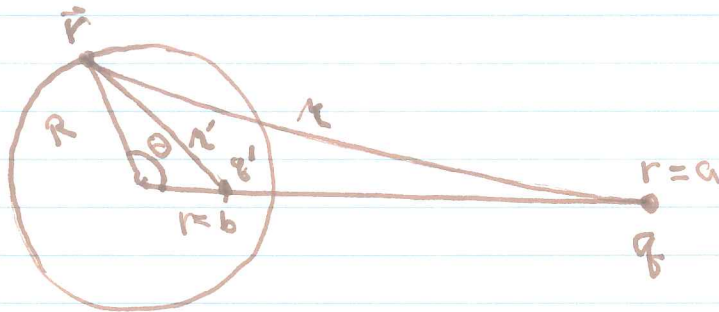
$$= \frac{\lambda_0}{2\pi\epsilon_0} \ln \left(\xi + \sqrt{s^2 + \xi^2} \right) \Big|_0^z = \frac{\lambda_0}{2\pi\epsilon_0} \ln \left(\frac{z + \sqrt{s^2 + z^2}}{s} \right)$$

GRIFFITHS §3.2.4
METHOD OF IMAGES IN SPHERICAL SYMMETRY

LORD KELVIN,
1848

①

CONDUCTING SPHERE

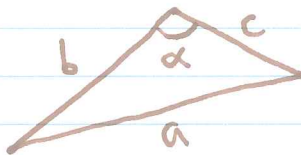


EXERCISE

CAN WE CHOOSE b AND q' SUCH THAT $V=0$ AT $\vec{r}=R\hat{r}$?

HINT: Law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$



ANSWER:

$$\begin{aligned} q' &= -\frac{R}{a} q \\ b &= R^2/a \end{aligned}$$

DISCUSS: ① WHAT IF WE HAD A SPHERICAL CAVITY, GROUNDED, WITH A CHARGE AT AN ARBITRARY LOCATION INSIDE?

② WHAT'S THE FORCE ON THE SPHERE?

$$F = \frac{1}{4\pi\epsilon_0} \frac{qq'}{(a-b)^2} = \frac{1}{4\pi\epsilon_0} \frac{q^2 R a}{(a^2 - R^2)^2} \quad \text{ALWAYS ATTRACTIVE.}$$

SKETCH $|F(a)|$

