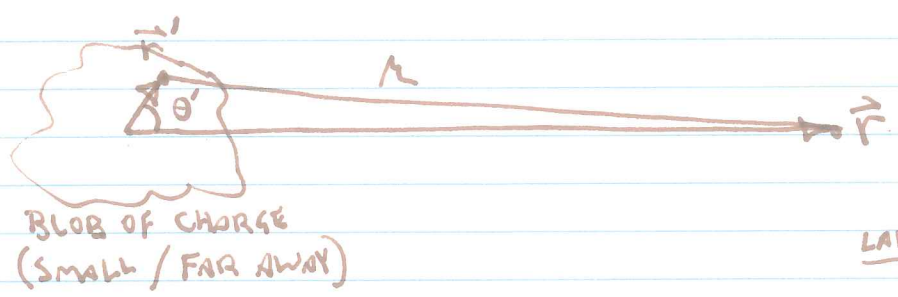


GRIFFITHS § 3.4

MULTIPOLE EXPANSION OF THE ELECTRIC POTENTIAL

TODAY

- SCHEDULE EXAM
- SCHEDULE REVIEW SESSION



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}') d\tau'}{R}, \quad \frac{1}{R} = \frac{1}{\sqrt{r'^2 + r^2 - 2rr' \cos\theta}}$$

IF $r' \ll r$, WE CAN APPROXIMATE $\frac{1}{R}$ USING A BINOMIAL EXPANSION.

ASIDE: MACLAURIN SERIES DERIVATION OF BINOMIAL EXPANSION

$$f(\epsilon) = (1 + \epsilon)^\alpha, \quad \epsilon < 1$$

MACLAURIN SERIES: (GENERAL CASE)

$$f(\epsilon) = f(0) + \epsilon f'(0) + \frac{1}{2} \epsilon^2 f''(0) + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) \epsilon^n}{n!}$$

FOR OUR BINOMIAL,

$$f = (1 + \epsilon)^\alpha; \quad f(0) = 1$$

$$f' = \alpha(1 + \epsilon)^{\alpha-1}; \quad f'(0) = \alpha$$

$$f'' = \alpha(\alpha-1)(1 + \epsilon)^{\alpha-2}; \quad f''(0) = \alpha(\alpha-1)$$

$$\therefore f^{(n)}(0) = \alpha(\alpha-1)\dots(\alpha-n+1)$$

n	$f^{(n)}(0)$ For $\alpha = -1/2$	$f^{(n)}/n!$
0	1	1
1	-1/2	-1/2
2	$(-1/2)(-3/2) = 3/4$	3/8
3	$(3/4)(-5/2) = -15/8$	$-15/48 = -5/16$

Finally, what we have is:

$$(1 + \epsilon)^{-1/2} = 1 - \frac{1}{2} \epsilon + \frac{3}{8} \epsilon^2 - \frac{5}{16} \epsilon^3 + \dots \quad (\text{Griffiths eg. 3.93})$$

FINALLY, EXPAND $1/\Lambda$...

$$\frac{1}{\Lambda} = \frac{1}{r} \left(1 - 2 \frac{r'}{r} \cos \theta' + \frac{r'^2}{r^2} \right)^{-1/2} = \frac{1}{r} (1 + \epsilon)^{-1/2},$$

$$\epsilon = -2 \frac{r'}{r} \cos \theta' + \frac{r'^2}{r^2}$$

$$\epsilon^2 = 4 \left(\frac{r'}{r} \right)^2 \cos^2 \theta' - 4 \left(\frac{r'}{r} \right)^3 \cos \theta' + \left(\frac{r'}{r} \right)^4$$

$$\epsilon^3 = -8 \left(\frac{r'}{r} \right)^3 \cos^3 \theta' + 4 \left(\frac{r'}{r} \right)^4 (2 \cos^2 \theta' + \cos^2 \theta') - 6 \left(\frac{r'}{r} \right)^5 \cos \theta' + \left(\frac{r'}{r} \right)^6.$$

$$\text{SO, } \frac{1}{\Lambda} = \frac{1}{r} \left[1 + \left\{ \left(\frac{r'}{r} \right)^2 - 2 \left(\frac{r'}{r} \right) \cos \theta' \right\} \left(\frac{-1}{2} \right) \right.$$

$$\left. + \left\{ 4 \left(\frac{r'}{r} \right)^2 \cos^2 \theta' - 4 \left(\frac{r'}{r} \right)^3 \cos \theta' + \left(\frac{r'}{r} \right)^4 \right\} \left(\frac{3}{8} \right) \right.$$

$$\left. + \left\{ -8 \left(\frac{r'}{r} \right)^3 \cos^3 \theta' + 4 \left(\frac{r'}{r} \right)^4 (2 \cos^2 \theta' + \cos^2 \theta') - 6 \left(\frac{r'}{r} \right)^5 \cos \theta' \right\} \left(\frac{-5}{16} \right) \right]$$

+ $\mathcal{O}(r'^4)$ ← THE $n=4$ TERM WOULD INCLUDE AN r'^4 , SO THE HIGHER ORDERS IN THIS EXPRESSION ARE INCOMPLETE!

COLLECTING TERMS BY ORDER IN r' ,

$$\frac{1}{\Lambda} = \frac{1}{r} \left[1 + \left(\frac{r'}{r} \right) \left\{ \cos \theta' \right\} + \left(\frac{r'}{r} \right)^2 \left\{ -\frac{1}{2} + \frac{3}{2} \cos^2 \theta' \right\} + \left(\frac{r'}{r} \right)^3 \left\{ -\frac{3}{2} \cos \theta' + \frac{5}{2} \cos^3 \theta' \right\} + \mathcal{O}(r'^4) \right]$$

THE TERMS IN $\{\}$ MAY BE RECOGNISED AS LEGENDRE POLYNOMIALS!

$$\therefore \frac{1}{\Lambda} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos \theta').$$

PUTTING IT ALL TOGETHER,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int r'^n P_n(\cos \theta') \rho(\vec{r}') d\tau'$$

THIS IS THE MULTIPOLE EXPANSION OF V .

NOTE THAT $d\tau' = r'^2 \sin \theta' dr' d\theta' d\phi'$.

I.E., WE HAVE TREATED \hat{r} LIKE \hat{z} .

COORDINATE INDEPENDENT EXPRESSION

MONOPOLE TERM (n=0)

$$V_{\text{MON}}(\vec{r}) = \frac{1}{4\pi\epsilon_0 r} \int \rho(\vec{r}') d\tau' = \frac{q}{4\pi\epsilon_0 r}$$

DISCUSS...

DIPOLE TERM (n=1)

$$V_{\text{DIP}}(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^2} \int r' \cos\theta' \rho(\vec{r}') d\tau'$$

NOTICE THAT $\vec{r}' \cdot \hat{r} = r' \cos\theta'$. THE ADVANTAGE OF THIS SUBSTITUTION IS THAT IT PUTS US BACK INTO FULLY GENERAL COORDINATES, SO WE CAN SEE EXPLICITLY HOW V DEPENDS ON θ AND ϕ ...

$$V_{\text{DIP}}(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^2} \hat{r} \cdot \int \vec{r}' \rho(\vec{r}') d\tau'$$

SO NOW $\int d\tau'$ DOES NOT HAVE TO USE OUR "ARTIFICIAL" θ' , WHICH WAS MEASURED FROM \hat{r} INSTEAD OF FROM \hat{z} . IN FACT, WE ARE FREE TO USE ANY COORDINATE SYSTEM, AS LONG AS IT HAS THE SAME ORIGIN AS \vec{r} .

WE DEFINE THE DIPOLE MOMENT

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau'$$

$$V_{\text{DIP}}(\vec{r}) = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2}$$

ASIDE
SINCE WE TREATED \hat{r} LIKE \hat{z} IN OUR DERIVATION, MERELY EVALUATING THE INTEGRALS DOES NOT GIVE US A GENERAL EXPRESSION FOR $V(\vec{r})$. BUT FOR SPECIAL CASES LIKE THE MONOPOLE & DIPOLE TERMS ON THIS PAGE, WE CAN EASILY INFER THE FORM OF SUCH A GENERAL $V(\vec{r})$. THAT'S WHAT THIS PAGE IS FOR!

EXERCISE ① WHAT IS THE DIPOLE MOMENT OF A COLLECTION OF POINT CHARGES?

② HOW DOES THE DIPOLE MOMENT CHANGE IF WE MOVE THE ORIGIN? DOES THIS AFFECT THE POTENTIAL? LOOK AT (A) SINGLE CHARGE, THEN (B) $\pm q$

① POINT CHARGES q_i @ \vec{r}'_i :

$$\rho(\vec{r}') = \sum_i q_i \delta^3(\vec{r}' - \vec{r}'_i)$$

$$\begin{aligned} \Rightarrow \vec{p} &= \int \vec{r}' \rho(\vec{r}') d\tau' = \int \vec{r}' \sum_i q_i \delta^3(\vec{r}' - \vec{r}'_i) d\tau' \\ &= \sum_i q_i \int \vec{r}' \delta^3(\vec{r}' - \vec{r}'_i) d\tau' \end{aligned}$$

$$\boxed{\vec{p} = \sum_i q_i \vec{r}'_i}$$

②(A) SUPPOSE q IS AT THE ORIGIN, $\vec{r}' = 0$.

I. $\vec{p} = q \vec{r}' = 0$.

NOW MOVE THE ORIGIN TO \vec{a} : $\vec{r}' = -\vec{a}$; SO

II. $\vec{p}^* = -q\vec{a}$

NOW LOOK AT THE POTENTIAL USING MULTIPOLE EXP. $n = 0, 1$.

I. $V = \frac{q}{4\pi\epsilon_0 r} + \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} = \frac{q}{4\pi\epsilon_0 r}$ (THE EXACT RESULT!)

II. MOVING THE ORIGIN TO $\vec{a} \Rightarrow \vec{r} \rightarrow \vec{r} - \vec{a}$, $\vec{p} \rightarrow -q\vec{a}$

$$V^* = \frac{q}{4\pi\epsilon_0 |\vec{r} - \vec{a}|} + \frac{-q\vec{a} \cdot \hat{r}}{4\pi\epsilon_0 |\vec{r} - \vec{a}|^2}$$

Q: HOW ACCURATE SHOULD THIS EXPANSION BE?

A: FIRST ORDER IN \vec{r}' , WHICH IS FIRST ORDER IN $\vec{a} \equiv [\alpha, \beta, \gamma]$

$$|\vec{r} - \vec{a}| = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2} \approx \sqrt{r^2 - 2\vec{a} \cdot \vec{r}}$$

$$\frac{1}{|\vec{r} - \vec{a}|} = \frac{1}{r(1 - \frac{2\vec{a} \cdot \vec{r}}{r})^{1/2}} = \frac{1 + (\vec{a} \cdot \vec{r})/r}{r}$$

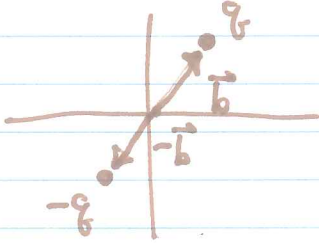
$$\frac{1}{|\vec{r} - \vec{a}|^2} = \frac{1 + 2(\vec{a} \cdot \vec{r})/r}{r^2} = \frac{1}{r^2} + \frac{2(\vec{a} \cdot \vec{r})/r}{r^2}$$

$$\therefore V^* = \frac{q}{4\pi\epsilon_0 r} \left[1 + \frac{\vec{a} \cdot \vec{r}}{r} - \left(\frac{\vec{a} \cdot \vec{r}}{r} + 2 \frac{(\vec{a} \cdot \vec{r})^2}{r^2} \right) \right] = \frac{q}{4\pi\epsilon_0 r} + \mathcal{O}(a^2)$$

OK!

2) (B) SUPPOSE WE HAVE THIS CHARGE CONFIGURATION:

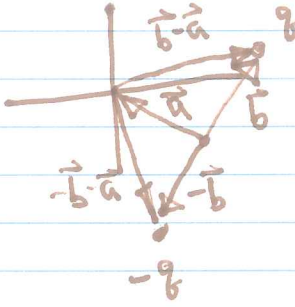
I.



$$\vec{p} = q\vec{b} + (-q)(-\vec{b}) = 2q\vec{b}$$

II. MOVE ORIGIN TO \vec{a}

SO $\vec{b} \rightarrow \vec{b} - \vec{a}$, $-\vec{b} \rightarrow -\vec{b} - \vec{a}$



$$\vec{p} \rightarrow q(\vec{b} - \vec{a}) + (-q)(-\vec{b} - \vec{a})$$

$$= 2q\vec{b}$$

UNCHANGED!

INTERPRETATION:

MONOPOLE MOMENT IS PROMOTED TO
 HAVE A DIPOLE MOMENT WHEN THE
 ORIGIN IS SHIFTED, BUT A "PURE"
 DIPOLE WITHOUT THE LEADING MONOPOLE
 IS INVARIANT.

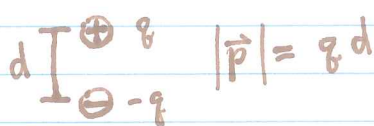
HIGHER n-POLE MOMENTS

n	NAME	MOMENT	TENSOR RANK
0	MONOPOLE	q, CHARGE, IS A SCALAR.	0
1	DIPOLE	\vec{p} IS A VECTOR $\in \mathbb{R}^3$	1
2	QUADRUPOLE	3x3 MATRIX	2
3	OCTOPOLE	3x3x3 TENSOR	3
⋮	⋮		

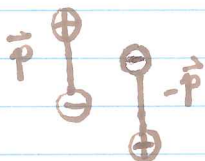
VISUALIZING THE MULTIPOLE EXPANSION



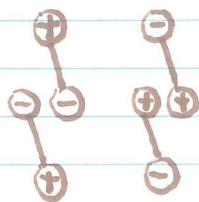
- MONOPOLE (NET CHARGE q)
- n = 0



- ZERO MONOPOLE MOMENT (ZERO NET CHARGE)
- LEADING TERM IS DIPOLE (n=1)



- ZERO NET DIPOLE MOMENT
- LEADING TERM IS QUADRUPOLE (n=2)



- ZERO QUADRUPOLE MOMENT (2 COPIES OF THE SAME QUADRUPOLE, WITH SIGNS REVERSED)
- LEADING TERM IS OCTOPOLE (n=3)

WE WON'T BE WORKING WITH $n \geq 2$ THIS SEMESTER, BUT IT IS WORTH HAVING A MENTAL PICTURE OF WHAT THE HIGHER ORDER TERMS ARE LIKE.

THE HIGHER ORDER THE LEADING TERM IN THE MULTIPOLE EXPANSION, THE MORE SYMMETRY THE CHARGE CONFIGURATION HAS.