

• ICE BREAKER — * NAME

* WHAT YOU LIKE TO DO ON WEEKENDS (BESIDES EGG HOMEWORK!)

* HAD APPLIED MATH OR MATH METHODS COURSE BEFORE?

• GO OVER SYLLABUS

• HAND OUT 1ST HW, DUE FRIDAY

• FIRST TOPIC:

- CH. 13, INTEGRAL TRANSFORMS
- FOURIER TRANSFORMS
- LAPLACE TRANSFORMS

APPLICATIONS OF FOURIER TRANSFORMS INCLUDE:

- DIFFRACTION THEORY, FOURIER OPTICS
- QUANTUM MECHANICAL SCATTERING THEORY
- SIGNAL - AND IMAGE - PROCESSING, DATA ANALYSIS
- SOLUTION OF ODEs
- " " PDEs
- " " INTEGRAL EQUATIONS

APPLICATION — QM WAVE FUNCTIONS

ENERGY EIGENSTATES (SOLN'S TO $H\psi_n = E_n\psi_n$)
OBEY AN ORTHONORMALITY CONSTRAINT:

$$\langle \psi_m | \psi_n \rangle \equiv \int_{-\infty}^{\infty} \psi_m^*(x) \psi_n(x) dx = \delta_{mn},$$

KRONECKER'S $\delta_{mn} \equiv \begin{cases} 1, & m=n \\ 0, & m \neq n \end{cases}$

INDEED, MOST QM WAVE FUNCTIONS, WHETHER
THEY ARE EIGENSTATES OR NOT, ARE NORMALIZED:

$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} \psi^*(x) \psi(x) dx = 1.$$

THIS DOESN'T WORK FOR MOMENTUM EIGENSTATES
OF A FREE PARTICLE:

$$\langle x | k \rangle = e^{ikx}, \quad P | k \rangle = \hbar k | k \rangle.$$

$$\langle k | k \rangle = \int_{-\infty}^{\infty} e^{ikx} e^{-ikx} dx = \int_{-\infty}^{\infty} dx \rightarrow \text{DIVERGES.}$$

$$\text{BUT! } \langle k | k' \rangle = \int_{-\infty}^{\infty} e^{-ikx} e^{+ik'x} dx$$

AS YOU'LL SHOW IN THE HW, THE INTEGRAL CAN BE DONE:

$$\langle k | k' \rangle = 2\pi \delta(k - k')$$

SO A PROPERLY NORMALIZED STATE SHOULD LOOK LIKE:

$$\langle x | k \rangle = \frac{1}{\sqrt{2\pi}} e^{ikx}.$$

(OPTIONAL)

HOW WOULD YOU NORMALIZE A STATE OF DEFINITE POSITION,

$$\psi_{x'}(x) = \delta(x-x') ?$$

$$\int_{-\infty}^{\infty} \psi_{x'}^*(x) \psi_{x''}(x) dx$$

$$= \int_{-\infty}^{\infty} \delta(x-x') \delta(x-x'') dx$$

TREAT AS δ

TREAT AS ANY OLD FUNCTION

$$\int_{-\infty}^{\infty} \psi_{x'}^* \psi_{x''} dx = \delta(x'-x'')$$

§ 13.1 FOURIER TRANSFORMS

DEFINITION:

$$\mathcal{F}[f(t)] = \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

NEGATIVE
↓

THE INTERPRETATION INVOLVES VECTORS IN A HILBERT SPACE. OUR BASIS ELEMENTS ARE

$$\frac{1}{\sqrt{2\pi}} e^{+i\omega t}$$

THE TRANSFORM "PROJECTS" $f(t)$ INTO ω -SPACE: $\tilde{f}(\omega) = \langle \omega | f \rangle$

ORTHONORMALITY OF THE FOURIER BASIS:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} e^{+i\omega' t} dt = \delta(\omega - \omega')$$

$$\int_{-\infty}^{\infty} \psi_{\omega}^*(t) \psi_{\omega'}(t) dt = \langle \omega | \omega' \rangle = \delta(\omega - \omega')$$

INVERSE TRANSFORM:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{+i\omega t} d\omega$$

↑ POSITIVE

(SOMETIMES CALLED FOURIER'S INVERSE THEOREM)

PROOF OF FOURIER'S INVERSE THEOREM:

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t') e^{-i\omega t'} dt' \right] e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t') e^{-i\omega t'} e^{i\omega t} dt' d\omega \end{aligned}$$

SWITCH ORDER OF INTEGRATION AND PULL $f(t')$ OUT OF $\int d\omega$:

$$= \int_{-\infty}^{\infty} f(t') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t'} e^{i\omega t} d\omega}_{\text{COMPARE TO NORMALIZATION CONDITION!}} dt'$$

$$= \int_{-\infty}^{\infty} f(t') \delta(t-t') dt'$$

$$= f(t) \quad \checkmark \quad \underline{\text{QED}}$$

COMMENT:

RHB CHOOSES A $\frac{1}{\sqrt{2\pi}}$ IN FRONT OF THE F.T.
THIS IS TO MAKE IT AS SYMMETRIC AS POSSIBLE
WITH THE INVERSE TRANSFORM.

F.T. PROPERTIES

LINEARITY

$$\mathcal{F}(f(t) + g(t)) = \tilde{f} + \tilde{g} \quad (\text{OBVIOUS})$$

DIFFERENTIATION

$$\boxed{\mathcal{F}[f'(t)] = i\omega \tilde{f}(\omega)} \quad (i)$$

PROOF:

$$\mathcal{F}[f'(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{df}{dt} dt$$

BY PARTS:

$$= \frac{1}{\sqrt{2\pi}} e^{-i\omega t} f(t) \Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -i\omega e^{-i\omega t} f(t) dt$$

E.G. ASSUME $\lim_{|t| \rightarrow \infty} f(t) = 0$

$$= i\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$$

$$= i\omega \tilde{f}(\omega) \quad \checkmark$$

INTEGRATION

$$\mathcal{F} \left[\int f(t) dt \right] = \frac{1}{i\omega} \tilde{f}(\omega) + \mathcal{F}(C)$$

WHAT'S $\mathcal{F}(C)$?

↑
CONSTANT OF INTEGRATION

$$\begin{aligned} \mathcal{F}(C) &= \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} dt \\ &= \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \underbrace{e^{i \cdot 0 \cdot t}}_1 dt \end{aligned}$$

COMPARE TO ORTHOGONALITY RELATION.

$$= \sqrt{2\pi} C \delta(\omega)$$

← SINCE C ARBITRARY, IGNORE!

SO

$$\mathcal{F} \left[\int f(t) dt \right] = \frac{1}{i\omega} \tilde{f}(\omega) + C \delta(\omega) \quad (ii)$$

ASIDE: IF YOU ARE AT ALL UNCOMFORTABLE WITH DIRAC'S δ , READ § 13.1.3.

CHECK: $\mathcal{F}(1) = \sqrt{2\pi} \delta(\omega)$
 $\Rightarrow \mathcal{F}^{-1}(\delta(\omega)) = \frac{1}{\sqrt{2\pi}}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \delta(\omega) dt = \frac{1}{\sqrt{2\pi}} e^{i \cdot 0 \cdot t} = \frac{1}{\sqrt{2\pi}} \checkmark$$

IN §§ 13.1.1-13.1.6 OF RHB OUTLINE:

FOURIER TRANSFORMS

- DEFINITION
- NORMALIZATION OF FOURIER "BASIS"
- CONNECTION WITH k -SPACE IN QM
- INVERSE
- PROPERTIES...

SO FAR, WE HAVE FOUND THE FOLLOWING PROPERTIES OF THE FOURIER TRANSFORM:

$$\text{LINEARITY: } \mathcal{F}(f(t) + g(t)) = \tilde{f}(\omega) + \tilde{g}(\omega)$$

$$\text{DIFFERENTIATION: } \mathcal{F}[f'(t)] = i\omega \tilde{f}(\omega)$$

$$\text{INTEGRATION: } \mathcal{F}\left[\int f(t) dt\right] = \frac{1}{i\omega} \tilde{f}(\omega) + C \delta(\omega)$$

EXERCISE:

WORK OUT EACH OF THESE

$$\mathcal{F}[f(at)] =$$

$$\mathcal{F}[f(a+t)] =$$

$$\mathcal{F}[e^{at} f(t)] =$$

REMAINING PROPERTIES (EASY TO SHOW):

SCALING:

$$\mathcal{F}[f(at)] = \frac{1}{a} \tilde{f}\left(\frac{\omega}{a}\right) \quad (iii)$$

TRANSLATION:

$$\mathcal{F}[f(a+t)] = e^{i\omega a} \tilde{f}(\omega) \quad (iv)$$

EXPONENTIAL MULTIPLICATION:

$$\mathcal{F}[e^{at} f(t)] = \tilde{f}(\omega + ia) \quad (v)$$

e.g. (iii): $\mathcal{F}(f(at)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(at) dt$

Let $t' = at \quad dt' = a dt$

$$= \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\left(\frac{\omega}{a}\right)t'} f(t') dt'$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega' t'} f(t') dt'$$

$$= \frac{1}{a} \tilde{f}(\omega') = \frac{1}{a} \tilde{f}\left(\frac{\omega}{a}\right) \quad \checkmark$$

EXAMPLE

FIND THE FOURIER TRANSFORM OF $\sin \omega_0 t$:

$$f(t) = \sin \omega_0 t = \frac{i}{2} (e^{-i\omega_0 t} - e^{i\omega_0 t})$$

(EULER'S THM).

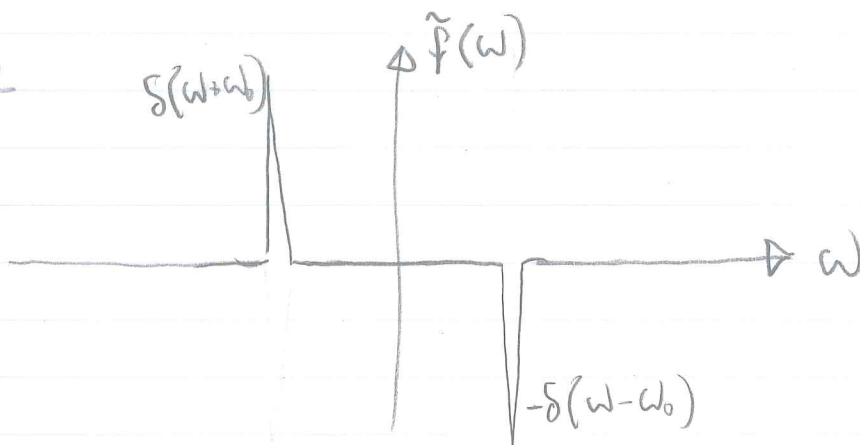
$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{i}{2} (e^{-i\omega_0 t} e^{-i\omega t} - e^{i\omega_0 t} e^{-i\omega t}) dt$$

USING THE NORMALIZATION RULE,

$$= \frac{i}{2\sqrt{2\pi}} [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \cdot 2\pi$$

$$= \frac{i}{2} \sqrt{2\pi} [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

DISCUSS

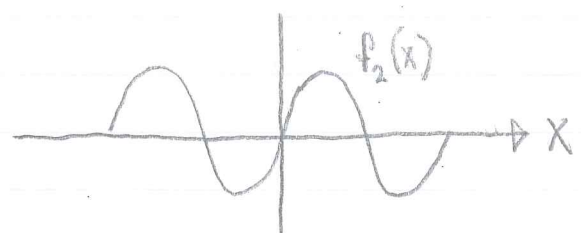


- LOCATIONS OF SPIKES
- SIGNS OF SPIKES
- CONSTANT FACTOR $\frac{i}{2}$
- CONSTANT FACTOR $\frac{1}{\sqrt{2\pi}}$

HINT: $\langle t | \omega \rangle = \frac{1}{\sqrt{2\pi}} e^{i\omega t}$

EXAMPLE — FOURIER TRANSFORM OF A BRIEF, SINUSOIDAL PULSE
SUPPOSE INSTEAD

$$f_n(x) = \begin{cases} \sin(\omega_0 t), & |t| < n\pi/\omega_0 \\ 0, & |t| > n\pi/\omega_0 \end{cases}$$



WHAT WILL $\tilde{f}_n(\omega)$ LOOK LIKE?

$$\begin{aligned} \tilde{f}_n(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_n(x) (\cos \omega t - i \sin \omega t) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-n\pi/\omega_0}^{n\pi/\omega_0} \sin \omega_0 t (\cos \omega t - i \sin \omega t) dt \end{aligned}$$

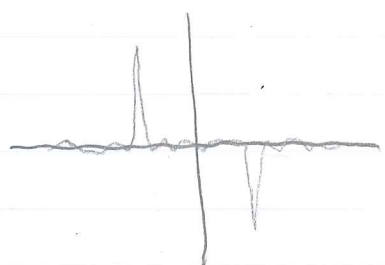
(ODD · EVEN / EVEN LIMITS)

Now,

$$\sin \omega_0 t \sin \omega t = \frac{1}{2} \cos(\omega - \omega_0)t - \frac{1}{2} \cos(\omega + \omega_0)t$$

$$\begin{aligned} \tilde{f}_n(\omega) &= \frac{-i}{2\sqrt{2\pi}} \left(\frac{\sin(\omega - \omega_0)t}{\omega - \omega_0} - \frac{\sin(\omega + \omega_0)t}{\omega + \omega_0} \right) \Big|_{-n\pi/\omega_0}^{n\pi/\omega_0} \\ &= \frac{-i}{\sqrt{2\pi}} \left(\frac{\sin(\omega - \omega_0) \frac{n\pi}{\omega_0}}{\omega - \omega_0} - \frac{\sin(\omega + \omega_0) \frac{n\pi}{\omega_0}}{\omega + \omega_0} \right) \end{aligned}$$

SEE GRAPHING CALCULATOR DEMO, FT example. of



MAIN POINTS

- $n=1$ CYCLE: POORLY DEFINED IN FREQUENCY SPACE
- $n \rightarrow \infty$: PEAKS GET TALLER & NARROWER.

SYMMETRIES OF \mathcal{F} ---

ANY REAL-VALUED FUNCTION $f(x)$ CAN BE WRITTEN UNIQUELY AS THE SUM OF AN EVEN PART + AN ODD PART:

$$f(x) = \underbrace{\frac{1}{2}(f(x) + f(-x))}_{\text{EVEN}} + \underbrace{\frac{1}{2}(f(x) - f(-x))}_{\text{ODD}}$$

ANY COMPLEX-VALUED FUNCTION $f(x)$ CAN BE WRITTEN THE SAME WAY. MOREOVER,

$$f(x) = \text{Re } f + i \text{Im } f$$

WHERE $\text{Re } f$ & $\text{Im } f$ ARE BOTH REAL-VALUED. SO...

$$f(x) = \underbrace{\text{Re}(f_{\text{EVEN}} + f_{\text{ODD}})}_{\text{Re } f} + i \underbrace{\text{Im}(f_{\text{EVEN}} + f_{\text{ODD}})}_{\text{Im } f}$$

NOW

$$\begin{aligned} \tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) (\cos \omega t - i \sin \omega t) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f_{\text{EVEN}}(t) \cos \omega t - i f_{\text{ODD}}(t) \sin \omega t] dt \\ &= \tilde{f}_{\text{EVEN}}(\omega) - i \tilde{f}_{\text{ODD}}(\omega) \end{aligned}$$

IN SUMMARY, $f(x)$ AND $\tilde{f}(\omega)$ EACH HAVE 4 PARTS:

