

NOW FOR AN EXAMPLE ...

A REMINDER ON HYPERBOLIC FUNCTIONS:

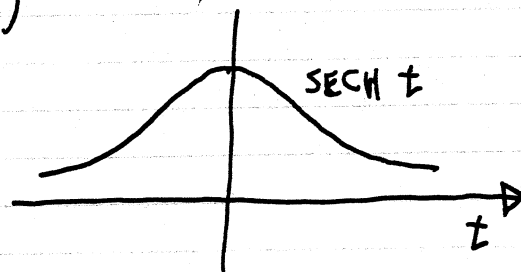
$$\cos t = \frac{1}{2} (e^{it} + e^{-it})$$

$$\cos(it) = \frac{1}{2} (e^{-t} + e^t) \equiv \cosh t, \text{ HYPERBOLIC COSINE}$$

$$\sinh t \equiv i \sin(it) = \frac{1}{2} (e^t - e^{-t})$$

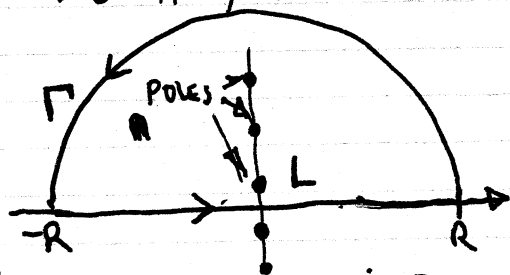
$$\tanh t \equiv \frac{\sinh t}{\cosh t} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

$$\operatorname{sech} t \equiv \frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}}$$



FIND  $\mathcal{F}(\operatorname{sech} t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2e^{-i\omega t}}{e^t + e^{-t}} dt$ .

- SINCE SECH IS EVEN, ITS FOURIER TRANSFORM IS EVEN.
- LET'S LOOK AT  $\omega < 0$ , SINCE THAT WAY THE INTEGRAL ON THE UPPER HALF PLANE ( $\Gamma$ ) WILL BE ZERO (I.E.  $e^{-i\omega z} \rightarrow 0$  IF  $y \gg 0$  AND  $\omega < 0$ ).



$$R \rightarrow \infty \\ \int_r \rightarrow 0$$

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{e^t + e^{-t}} dt = \oint \frac{e^{-i\omega z}}{e^z + e^{-z}} dz, \quad \omega < 0.$$

THE POLES ARE AT:  $e^z = -e^{-z} = e^{i\pi(1+2m)} e^{-z}$   
 $\therefore z_m = \cancel{i\pi} i\pi(m + \frac{1}{2}), \quad m \in \mathbb{Z}$

ARE THEY SIMPLE POLES? IF SO, THE FOLLOWING LIMIT WILL BE FINITE, AND WILL TURN OUT TO BE THE RESIDUE:

$$\lim_{z \rightarrow z_m} \frac{(z - z_m) e^{-i\omega z}}{e^z + e^{-z}} \xrightarrow{\text{L'HOPITAL}} \lim_{z \rightarrow z_m} \frac{e^{-i\omega z}}{e^z - e^{-z}} = \frac{e^{-i\omega z_m}}{e^{z_m} - e^{-z_m}} = \frac{e^{-i\omega z_m}}{2e^{z_m}}$$

(YES!)  $R_m = \frac{e^{-i\omega z_m}}{2e^{z_m}}$

Now, recall  $z_m = i\pi(m + \frac{1}{2}), m \in \mathbb{Z}$ .

$$e^{z_m} = e^{i\pi/2} e^{i\pi m} = i(-1)^m.$$

SO THE RESIDUES ARE:

$$R_m = \frac{-i}{2} (-1)^m e^{(m+\frac{1}{2})\pi\omega}, \quad \underline{\omega < 0}.$$

AND THE INTEGRAL IS:

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{e^t + e^{-t}} dt = 2\pi i \sum R_m = \pi \sum_{m=0}^{\infty} (-1)^m e^{(m+\frac{1}{2})\pi\omega}.$$

SUM ONLY OVER POLES IN THE UPPER HALF PLANE.

LET  $x = e^{\frac{1}{2}\pi\omega}$ . THEN THE SERIES IS:

$$\pi x (1 - x^2 + x^4 - x^6 + \dots)$$

NOW, CONSIDER THE FOLLOWING ~~EXACT~~ BINOMIAL EXPANSION:

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots \quad * \quad \text{EVIDENTLY,}$$

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{e^t + e^{-t}} dt = \pi \frac{e^{\pi\omega/2}}{1 + e^{\pi\omega}} = \frac{\pi}{2} \text{SECH}(\pi\omega/2)$$

SO,

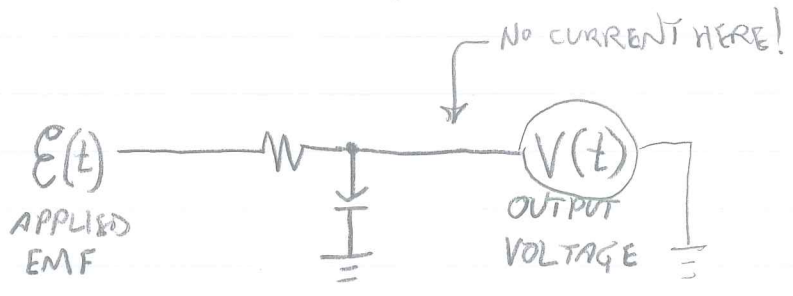
$$\boxed{\mathcal{F}(\text{SECH } t) = \sqrt{\frac{\pi}{2}} \text{SECH}(\pi\omega/2)}$$

THE SECH IS SELF-FOURIER!

Find b such that  $\mathcal{F}[\text{sech}(bt)] = \text{sech}(b\omega)$ .

$$* (1+a)^\alpha = 1 + \alpha a + \frac{\alpha(\alpha-1)}{2!} a^2 + \dots = \sum_{k=0}^{\infty} \binom{\alpha}{k} a^k$$

APPLICATION : SOLUTION OF <sup>INHOMOGENEOUS</sup> LINEAR ODE'S (NAIVE EXAMPLE - CORRECT LATER!)



WHAT'S THE ODE FOR  $V(t)$  ?

$$E - V = \dot{q}R; \quad q = CV \Rightarrow \dot{q} = C\dot{V}$$

$$E - V = RC\dot{V}$$

$$RC\dot{V} + V = E(t)$$

NOW, FOURIER TRANSFORM BOTH SIDES:

$$\tilde{V}(\omega) [i\omega RC + 1] = \tilde{E}(\omega)$$

$$\tilde{V}(\omega) = \frac{1}{1 + i\omega RC} \tilde{E}(\omega)$$

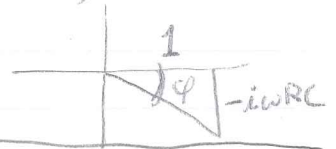
IN FOURIER SPACE, THE ODE IS A SIMPLE, ALGEBRAIC EQN,

THE QUANTITY

$$H(\omega) = \frac{1}{1 + i\omega RC} = \frac{1 - i\omega RC}{1 + (\omega RC)^2} = \overset{\text{MTF}}{\downarrow} A(\omega) e^{i\phi(\omega)}$$

PHASE XFER FXN (PTF)

IS THE FREQUENCY RESPONSE OF THE SYSTEM. IT HAS A MAGNITUDE (GAIN OR ATTENUATION) AND A PHASE:



$$A = \frac{1}{\sqrt{1 + (\omega RC)^2}}, \quad \phi = -\tan^{-1} \omega RC$$

THE SOLUTION FOR  $V(t)$  IS:

$$V(t) = \mathcal{F}^{-1} \left[ \frac{\tilde{E}(\omega)}{1+i\omega RC} \right]$$

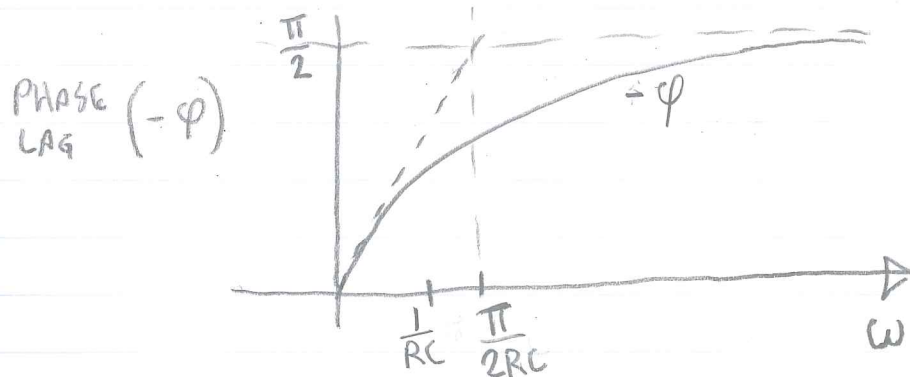
NOW, IF  $E(t) = E_0 e^{i\omega_f t}$ , THINGS BECOME VERY STRAIGHTFORWARD:

$$\tilde{E}(\omega) = \sqrt{2\pi} E_0 \delta(\omega - \omega_f)$$

$$\begin{aligned}
 V(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \frac{\sqrt{2\pi} E_0 \delta(\omega - \omega_f)}{1+i\omega RC} d\omega \\
 &= \frac{E_0 e^{i\omega_f t}}{1+i\omega RC} = E_0 A e^{i(\omega t + \phi)} \\
 &\hspace{15em} \underbrace{\hspace{10em}}_{R(\omega)}
 \end{aligned}$$

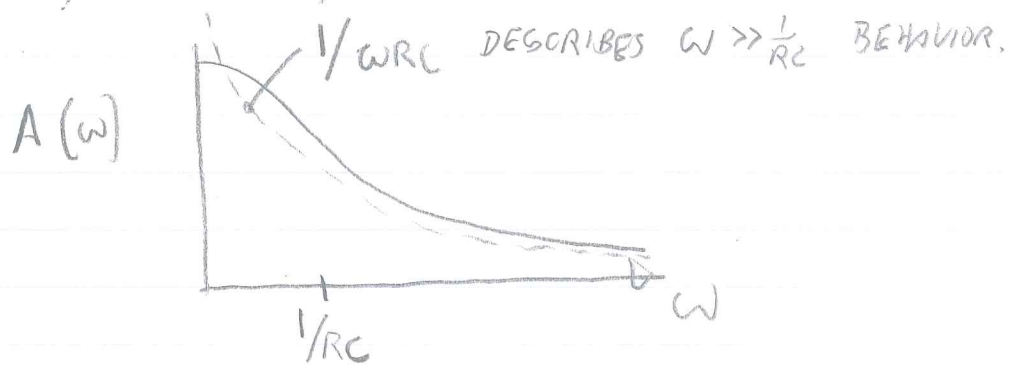
THAT IS,  $V = E \underbrace{A e^{i\phi}}_{R(\omega)}$  IF  $E(t)$  IS HARMONIC,

→ IN THIS CASE, THE PHASE  $\phi = -\tan^{-1} \omega RC < 0$ , WHICH MEANS THAT  $V(t)$  LAGS  $E(t)$ .



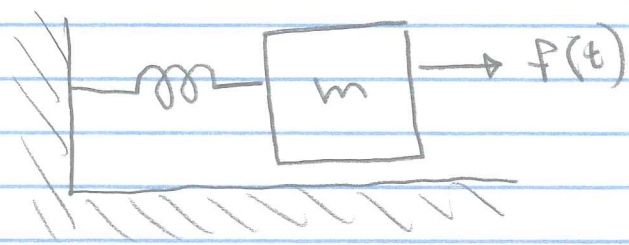
THE FILTER SUPPRESSES HIGH FREQUENCIES, WITH RESPONSE:

$$A = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \quad (\text{LORENTZIAN})$$



YOU'LL SOLVE AN ODE IN EXERCISE 13.5.

IN HW AND CLASS EXAMPLES, WE HAVE BEEN IGNORING THE HOMOGENEOUS SOLN. OF THE ODE.



$$m\ddot{x} = -kx + F(t)$$

$$\tilde{x}(\omega) = \frac{\tilde{F}(\omega)}{k - m\omega^2}$$

THIS SEEMS TO IMPLY THAT IF  $F=0$ ,  $X=0$ .

CONSIDER:

$$m\ddot{x} + kx = F(t) = 0$$

$$(k - m\omega^2)\tilde{x}(\omega) = 0$$

$$\tilde{x} = 0 \text{ OR } k = m\omega^2$$

$$\omega = \pm\sqrt{k/m} \equiv \omega_0$$

$$\text{i.e. } x = A e^{i\omega_0 t} + B e^{-i\omega_0 t}$$

SINCE THE EQUATION IS LINEAR,

$$x(\tilde{t}) = \mathcal{F}^{-1} \left( \frac{\tilde{F}(\omega)}{k - m\omega^2} \right) + A e^{i\omega_0 t} + B e^{-i\omega_0 t}$$

ADDING THE HOMOGENEOUS SOLUTION ALLOWS US TO HAVE ANY INITIAL CONDITION, INDEPENDENT OF FORCING.

CONSEQUENCE: THE FULL SOLN. MUST INCLUDE THE HOMOGENEOUS PIECE:

$$\tilde{x}(\omega) = \frac{\tilde{f}(\omega)}{k - m\omega^2} + A \delta(\omega - \sqrt{k/m}) + B \delta(\omega + \sqrt{k/m})$$

Q: WHAT IF  $f(t) = \cos \sqrt{\frac{k}{m}} t$  ?

Q: WHAT DOES THIS IMPLY ABOUT THE SOLUTION TO OUR RC CIRCUIT?