

# FOURIER TRANSFORMS SO FAR

- o DEFINITION
- o NORMALIZATION (AND CONNECTION WITH RM)
- o INVERSE
- o PROPERTIES

## LINEARITY

- i DIFFERENTIATION
- ii INTEGRATION
- iii SCALING
- iv TRANSLATION
- v EXPONENTIAL MULTIPLICATION

- o (EXAMPLE  $f \rightarrow \text{isofrom}$ )
- o SYMMETRIES — ODD & EVEN FUNCTIONS
- o SOLUTION OF LINEAR ODE'S (2 EXAMPLES)

## TODAY

~~FINISH WITH FRIDAY'S LAST EXAMPLE~~

HW?

## CONVOLUTION

### CONV THEOREM

(EXAMPLE: HOW TO BUILD A GAUSSIAN FROM ANYTHING)

### CROSS-CORRELATION

# CONVOLUTION (§ 13.1.7)

①

$$f(t) * g(t) \equiv \int_{-\infty}^{\infty} f(x) g(t-x) dx$$

$$g * f = \int_{-\infty}^{\infty} g(x) f(t-x) dx$$

$$\text{LET } z = t - x$$

$$dz = -dx$$

$$g * f = - \int_{z=\infty}^{-\infty} g(t-z) f(z) dz$$

$$= \int_{-\infty}^{\infty} f(z) g(t-z) dz$$

$$\therefore \boxed{g * f = f * g}$$

NOTE THAT CONVOLUTION IS EXACTLY WHAT WE DO WITH GREEN'S FUNCTIONS:

**WILL**  
GIVEN  $\mathcal{L} u(x) = f(x)$

$$\mathcal{L} G(x-x_0) = \delta(x-x_0)$$

RECALL  $u(x) = \int_D G(x-x_0) f(x_0) dx_0 = f * G.$

EXAMPLE

AN AMPLIFIER HAS A PULSE RESPONSE

$$P(t) = \begin{cases} At e^{-t/\tau}, & t > 0 \\ 0, & t < 0 \end{cases}$$

SO THE RESPONSE TO AN INPUT SIGNAL  $S(t)$  IS:

$$R(t) = \int_{-\infty}^{\infty} S(t') P(t-t') dt'$$

a) FIND THE GAIN AT DC: ( $g_0$ )

IF  $S(t) = 1$ ,  $R(t) = g_0$

$$g_0 = \int_{-\infty}^{\infty} P(t-t') dt'$$

$$= \int_0^{\infty} Ate^{-t/\tau} dt$$

$$\left. \begin{aligned} u &= t & dv &= e^{-t/\tau} dt \\ du &= dt & v &= -\tau e^{-t/\tau} \end{aligned} \right\} \int u dv = uv - \int v du$$

$$g_0 = -At\tau e^{-t/\tau} \Big|_0^{\infty} + A \int_0^{\infty} \tau e^{-t/\tau} dt$$

$$= 0 - A\tau^2 e^{-t/\tau} \Big|_0^{\infty} = A\tau^2$$

$g_0 = A\tau^2$

SKIP FOR NOW.  
WE CAN LOOK BACK AT THIS AFTER WE'VE DONE GREEN'S FUNCTIONS.

b) WHAT IS THE MINIMUM TIME BETWEEN RESOLVED PULSES?

NOW, WHAT HAPPENS IF WE FOURIER TRANSFORM THE CONVOLUTION?

$$f * g = \int_{-\infty}^{\infty} f(t') g(t-t') dt'$$

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' e^{-i\omega t} f(t') g(t-t') \quad \times$$

$$= \int_{-\infty}^{\infty} dt' f(t') \mathcal{F}[g(t-t')]$$

$$= \int_{-\infty}^{\infty} dt' f(t') e^{-i\omega t'} \tilde{g}(\omega)$$

$$\boxed{\mathcal{F}(f * g) = \sqrt{2\pi} \tilde{f}(\omega) \tilde{g}(\omega)} \quad (13.38)$$

CONVOLUTION THEOREM.

SO CONVOLUTION IN REAL SPACE IS  
MULTIPLICATION IN FREQUENCY SPACE!

SIMILARLY,  $\frac{1}{\sqrt{2\pi}} \tilde{f}(\omega) * \tilde{g}(\omega) = \mathcal{F}(f(t)g(t))$ . (13.39)

$$\text{l.e. } \boxed{\mathcal{F}^{-1}(\tilde{f} * \tilde{g}) = \sqrt{2\pi} fg}$$

### IMPLICATION FOR GREEN'S FUNCTIONS:

SUPPOSE AGAIN THAT

$$\mathcal{L} u(t) = f(t)$$

AND WE HAVE A GREEN'S FUNCTION  $G(t-t_0)$ , SO

$$u(t) = \int_{-\infty}^{\infty} G(t-t_0) f(t_0) dt_0 = f * G.$$

$$\text{THEN } \tilde{u}(\omega) = \tilde{f}(\omega) \tilde{G}(\omega)$$

$$\text{AND } \tilde{G}(\omega) = \frac{\tilde{u}}{\tilde{f}} \begin{matrix} \leftarrow \text{SYSTEM RESPONSE} \\ \leftarrow \text{FORCING} \end{matrix}$$

IN OTHER WORDS,  $\tilde{G}(\omega)$  IS THE FREQUENCY RESPONSE!

SO GREEN'S FUNCTIONS AND THE FOURIER TRANSFORM METHOD FOR SOLVING DIFFERENTIAL EQUATIONS ARE ACTUALLY HIGHLY RELATED:

→ G.F. ASKS RESPONSE TO  $\delta(t-t_0)$

→ F.T. ASKS RESPONSE TO  $\delta(\omega-\omega_0)$



①

## EXAMPLE: USEFULNESS OF CONVOLUTION IN PROBABILITY THEORY

SUPPOSE WE HAVE TWO STATISTICALLY INDEPENDENT RANDOM VARIABLES,

$\alpha_1, \alpha_2$   
WITH PDF'S  $P_1(\alpha_1), P_2(\alpha_2)$ .

LET  $X = \alpha_1 + \alpha_2$ . WHAT IS  $P(X)$ ?

THE PROBABILITY OF GETTING  $X$  FOR A SPECIFIC  $\alpha_1$  IS:

$$\begin{aligned} P(X | \alpha_1) &= P_2(\alpha_2) \quad \text{WHERE } X = \alpha_1 + \alpha_2, \text{ I.E.} \\ &= P_2(X - \alpha_1). \end{aligned}$$

OF COURSE,  $\alpha_1$  CAN BE ANYTHING, AND IS DISTRIBUTED AS GIVEN BY  $P_1$ . SO THE PROBABILITY OF GETTING  $X$  REGARDLESS OF THE VALUE OF  $\alpha_1$  IS:

$$\begin{aligned} P(X) &= \int_{-\infty}^{\infty} P(X | \alpha_1) P_1(\alpha_1) d\alpha_1 \\ &= \int_{-\infty}^{\infty} P_1(\alpha_1) P_2(X - \alpha_1) d\alpha_1 \end{aligned}$$

$$\boxed{P(X) = P_1(X) * P_2(X)}$$

EXAMPLE: CENTRAL LIMIT THEOREM (RHB § 36.10)

THEOREM

SUPPOSE  $\alpha_i, i = 1, 2, \dots, N$ , ARE INDEPENDENT RANDOM VARIABLES, WITH PDF'S  $f_i(\alpha_i)$ , WITH MEAN  $\mu_i$  AND VARIANCE  $\sigma_i^2$ .

THE RANDOM VARIABLE  $X = \sum_i \alpha_i$ , HAS THE PROPERTIES:

- i.  $\langle X \rangle = \sum_i \mu_i$
- ii. VARIANCE  $\langle [X - \langle X \rangle]^2 \rangle = \sum_i \sigma_i^2$

iii. AS  $N \rightarrow \infty$ ,  $P(x)$  TENDS TO A GAUSSIAN WITH THE ~~MEAN~~ MEAN & VARIANCE GIVEN ABOVE.

IN WHAT FOLLOWS, I WANT THE  $N \rightarrow \infty$  BEHAVIOR AS "NICE" AS POSSIBLE. LET'S SPECIALIZE TO A SIMPLER CASE:

- LET  $f_i(\alpha_i) = f(\alpha_i)$  (ALL  $N$  VARIABLES WITH THE SAME DISTRO)

- CHOOSE  $f(\alpha_i)$  SO THAT  $\mu = 0$  ( $\langle \alpha \rangle = 0$ )

- CHOOSE  $f(\alpha_i)$  SO THAT  $\sigma^2 = \frac{1}{N}$ , AND THUS

$$\langle [X - \langle X \rangle]^2 \rangle = \langle X^2 \rangle = \sum_{i=1}^N \frac{1}{N} = 1$$

WE WILL THEN EXPECT TO FIND

$$\lim_{N \rightarrow \infty} P(x) = \overbrace{f * f * f * \dots * f}^{N \text{ TIMES}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

↑  
CONVOLUTION



OR, USING THE CONVOLUTION THEOREM, § 13.1.1

lim\_{N to infinity} [tilde{f}(k)]^N = 1/sqrt(2pi) \* e^{-k^2/2}

THIS IS A MAGICAL PROPERTY, IF TRUE!

- PICK AN ARBITRARY f(x) > 0, integral f dx = 1
- TWEAK SO THAT mu = 0 AND sigma = 1/sqrt(N)

THEN tilde{f}^N(k) -> 1/sqrt(2pi) \* e^{-k^2/2} !!!

Q: -> WHAT ARE THE REQUIREMENTS ON f, f(x) > 0, integral f dx = 1, SO THAT THIS WORKS?

Q: -> WHAT ARE THE REQUIREMENTS ON f SO THAT

tilde{f}^N(k) -> A \* e^{-(k-k\_bar)^2 / 2\*sigma^2}

FOR SOME A, k\_bar, sigma



FOR FUN, LET'S TRY OUT

$$P_1(\alpha) = \frac{1}{2a} e^{-|\alpha|/a}$$

(Q: IS THE NORMALIZATION CORRECT?)

$$\tilde{P}_1(k) = \frac{1}{(1+a^2 k^2)}$$

CONVOLVING  $N$  COPIES OF  $P(\alpha)$  LEADS TO  $P_N(x)$ ;

$$\tilde{P}_N(k) = \tilde{P}_1^N = \frac{1}{(1+a^2 k^2)^N}$$

$$= \frac{1}{1 + Na^2 k^2 + \frac{N(N-1)}{2} a^4 k^4 + \frac{N(N-1)(N-2)}{6} a^6 k^6 + \dots}$$

TO KEEP  $\sigma$  FINITE AS  $N \rightarrow \infty$ , LET  $a = \frac{1}{\sqrt{N}}$ .

NOW,

$$\begin{aligned} \tilde{P}_N^N &= \frac{1}{1+k^2 + \frac{k^4}{2} + \frac{k^6}{6} + \frac{k^8}{24} + \dots} \\ &= \frac{1}{e^{k^2}} = e^{-k^2} \end{aligned}$$

Q: SUPPOSE I CHOOSE  $P_1(\alpha) = \frac{A}{1+a^2 \alpha^2}$ ,  
so  $\tilde{P}_1(k) \propto e^{-k/a}$ .

IS  $\tilde{P}_1^N$  A GAUSSIAN? EXPLAIN!