

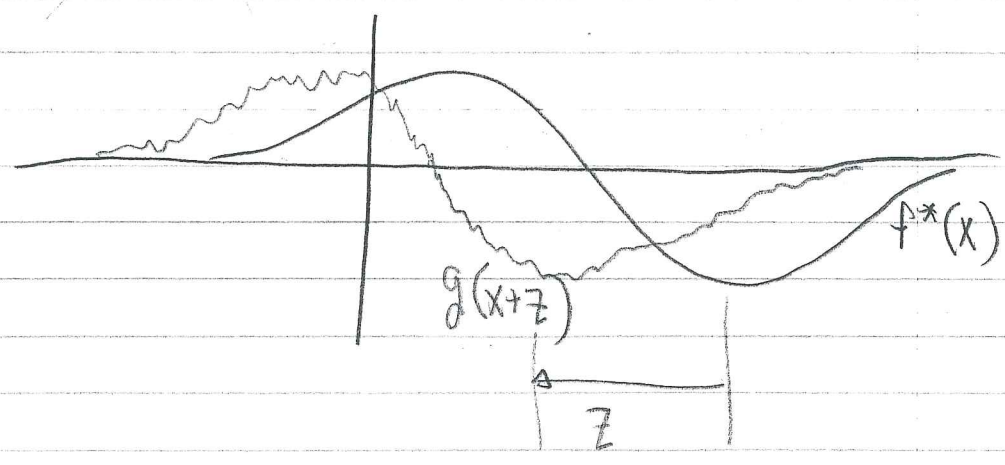
### CROSS-CORRELATION

→ HOW ALIKE ARE  $f(x)$  AND  $g(x)$ ?

$$f \otimes g \equiv \int_{-\infty}^{\infty} f^*(x) g(x+z) dx = C(z)$$

NOTE DIFFERENCE FROM CONVOLUTION!

• E.G. COMPARING DATA TO THEORY:



• CAN BE USED TO CO-REGISTER TWO COMMENSURATE DATASETS (MAXIMIZE  $C(z)$ ).

• CORRELATION & CONVOLUTION ARE INTERCHANGEABLE FOR THE SPECIAL CASE OF EVEN FUNCTIONS  $f$  &  $g$ .

PROPERTIES:

$$[f \otimes g](z) = [g \otimes f]^*(-z) \quad \leftarrow \text{UNLIKE } f * g = g * f$$

$$f \otimes (g+h) = f \otimes g + f \otimes h \quad \leftarrow \text{LIKE } f * (g+h)$$

WIENER-KINCHIN THEOREM:

$$\mathcal{F}[f \otimes g] = \sqrt{2\pi} [\tilde{f}(k)]^* \tilde{g}(k) \quad (13.42)$$

→ PROOF IN TEXT.

SIMILARLY,

$$\mathcal{F}[f^*(x) g(x)] = \frac{1}{\sqrt{2\pi}} \tilde{f} \otimes \tilde{g}$$

## AUTO CORRELATION FUNCTION OF $f(x)$ :

$$a(z) \equiv \int_{-\infty}^{\infty} f^*(x) f(x+z) dx = f \otimes f$$

### PROPERTIES:

$$a(z) \leq a(0)$$

CONSIDER THE FOURIER TRANSFORM OF THE AUTOCORRELATION:

$$a(z) = \mathcal{F}^{-1}(\tilde{a}(k)) \quad \text{BETTER:} \quad \tilde{a} = \mathcal{F}(a) = \mathcal{F}(f \otimes f)$$

$$= \sqrt{2\pi} \tilde{f}^* \tilde{f}$$

USING W-K THEOREM,

$$= \mathcal{F}^{-1}(\sqrt{2\pi} \tilde{f}^* \tilde{f})$$

$$\mathcal{F}(a) = \sqrt{2\pi} |\tilde{f}|^2$$

$$\text{SO, } \mathcal{F}(a(z)) = \sqrt{2\pi} \tilde{f}^* \tilde{f}$$

$$= \sqrt{2\pi} |\tilde{f}(k)|^2$$

POWER SPECTRUM

"THE FOURIER TRANSFORM OF THE AUTOCORRELATION IS THE POWER SPECTRUM".

NOTE THAT THE NORMALIZATION OF THE POWER SPECTRUM VARIES ACCORDING TO THE F.T. NORMALIZATION!

- THE AUTOCORRELATION AND THE POWER SPECTRUM BOTH HAVE LESS INFORMATION THAN THE SIGNAL  $f(x)$ : PHASE INFORMATION HAS BEEN LOST.

- THE AUTOCORRELATION CONTAINS EXACTLY THE SAME INFORMATION THAT IS IN THE POWER SPECTRUM;

$$f(x) \text{ IS TO } \tilde{f}(k)$$

AS

$$a(x) \text{ IS TO } \tilde{f}^* \tilde{f}.$$

- PERIODICITIES IN  $f(x)$  APPEAR AS:

- A PEAK IN  $\tilde{f}^* \tilde{f}$

- A TRAIN OF PULSES IN  $a(x)$

- $\tilde{f}^* \tilde{f} \geq 0$ , BUT  $a(x)$  CAN BE POSITIVE OR NEGATIVE.

# PARSEVAL'S THEOREM (§ 13.1.9)

THE CROSS-CORRELATION OF  $f$  AND  $g$  IS:

$$C(z) \equiv \int_{-\infty}^{\infty} f^*(x) g(x+z) dx$$

W-K THEOREM:

$$\tilde{C}(k) = \sqrt{2\pi} \tilde{f}^*(k) \tilde{g}(k)$$

FOURIER<sup>-1</sup> TRANSFORM AND ...

$$C(z) \equiv \int_{-\infty}^{\infty} f^*(x) g(x+z) dx = \int_{-\infty}^{\infty} \tilde{f}^*(k) \tilde{g}(k) e^{ikz} dk$$

SETTING  $z=0$ ,

$$\int_{-\infty}^{\infty} f^*(x) g(x) dx = \int_{-\infty}^{\infty} \tilde{f}^*(k) \tilde{g}(k) dk \quad (13.45)$$

("MULTIPLICATION THEOREM")

FOR  $f=g$ ,

$$\int_{-\infty}^{\infty} f^* f dx = \int_{-\infty}^{\infty} \tilde{f}^* \tilde{f} dk, \quad (13.46)$$

PARSEVAL'S THEOREM.

NOTE THAT IF THE  $\mathcal{F}$  IS NORMALIZED DIFFERENTLY, THERE CAN BE FACTORS OF  $\sqrt{2\pi}$  ON EITHER SIDE OF THIS FORMULA. BUT WE'VE CHOSEN THE MOST COMMON NORMALIZATION

### DATA ANALYSIS APPLICATION : WHITE NOISE

CONSIDER A SIGNAL  $s(t)$  IN WHICH THERE IS NO RELATIONSHIP BETWEEN WHAT HAPPENS AT TIME  $t$  AND AT TIME  $t + \Delta t$ . THIS IS UNCORRELATED, OR WHITE, NOISE.  
(E.G. ADDITIVE GAUSSIAN NOISE)

UNCORRELATED NOISE HAS, BY DEFINITION, NO PERIODICITIES. EVEN THE SLIGHTEST SHIFT IN  $s(t)$  SHOWS NO CORRELATION:

$$\langle a(\Delta t) \rangle = 0, \quad \Delta t \neq 0$$

$\rightarrow \langle a^2(\Delta t) \rangle \approx N_{\text{POINTS}}$

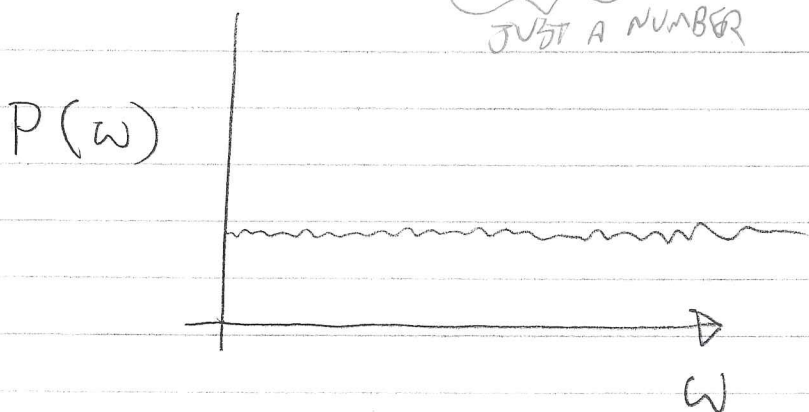
BUT IF  $\Delta t = 0$ , OF COURSE WE FIND THAT  $s(t)$  IS PERFECTLY CORRELATED WITH ITSELF.

$$\text{THUS, } a(\Delta t) = \delta(\Delta t) \int_{-\infty}^{\infty} s^*(t) s(t) dt$$

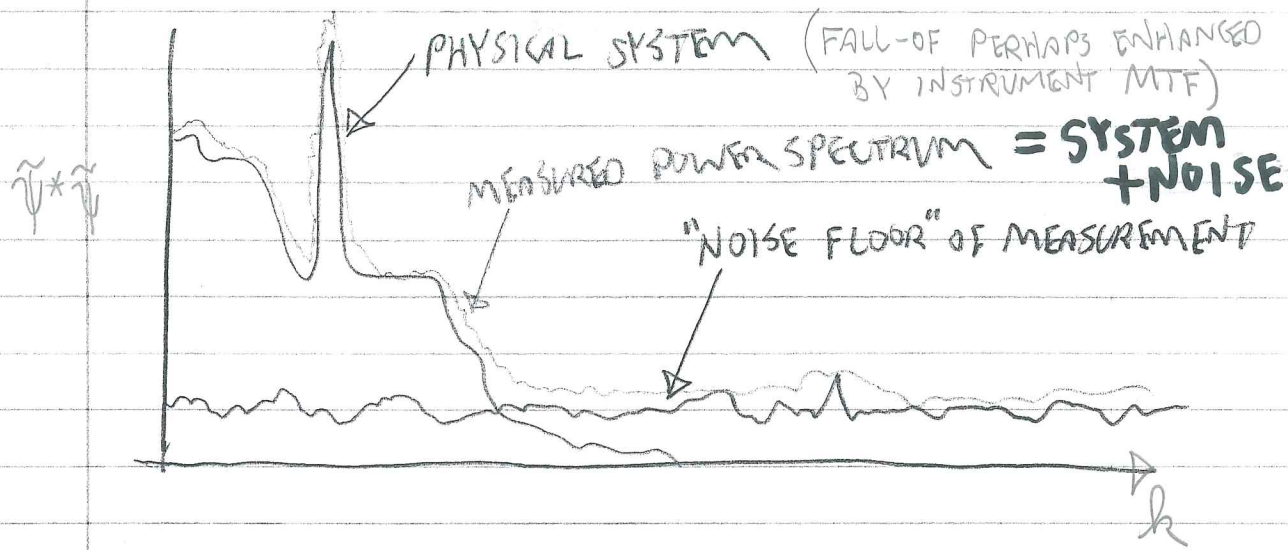
NORMALIZED USING PARSEVAL'S THM.

THE POWER SPECTRUM IS THUS

$$\mathcal{F} (a(\Delta t)) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} s^* s dt}_{\text{JUST A NUMBER}} \text{ --- FLAT ("WHITE")}$$



PHYSICAL SIGNALS USUALLY HAVE A DECREASING POWER SPECTRUM:



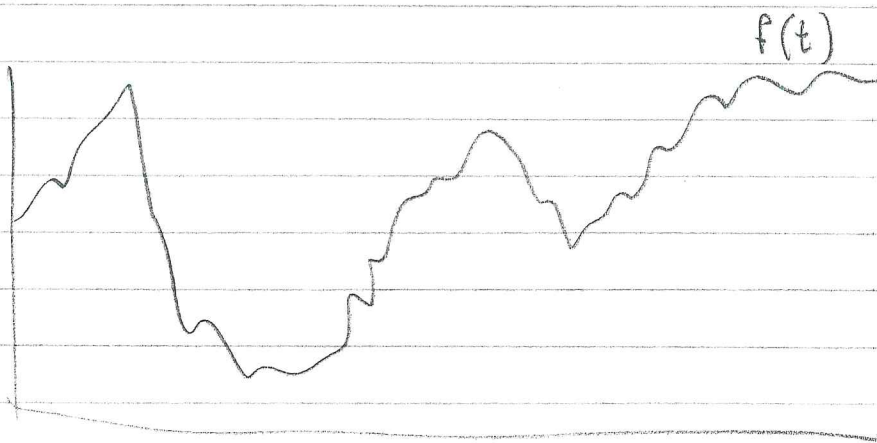
THUS, IT IS QUITE NATURAL TO FILTER DATA TO REMOVE NOISE BY SMOOTHING OVER FINE DETAILS (REMOVING HIGH SPATIAL FREQUENCIES)

- "GRAINY" PHOTOGRAPHS
- "SNOW" ON THE TV
- "STATIC" ON THE RADIO
- "HISS" ON CASSETTE TAPES

WE NOTICE THE NOISE AT HIGH FREQUENCIES, BECAUSE THAT IS WHERE IT DOMINATES OVER THE SIGNAL.

# DATA ANALYSIS APPLICATION: SMOOTHING

OFTEN, WHEN DATA IS NOISY,



WE "SMOOTH" THE DATA WITH A BOXCAR:

$$s(t) = \int_{-\infty}^{\infty} \Pi(t-t') f(t') dt'$$

$$\text{WHERE } \Pi(t) \equiv \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \text{ELSE.} \end{cases}$$

- THIS "MOVING AVERAGE" CAN BE ACCOMPLISHED CONVENIENTLY IN FOURIER SPACE:

$$S(\omega) = f(\omega) \tilde{\Pi}(\omega)$$

- THE FFT MAKES IT VERY FAST TO MOVE BETWEEN THE TIME DOMAIN AND THE FREQUENCY DOMAIN, SO THIS APPROACH IS COMPUTATIONALLY CHEAP.

$$s(t) = \mathcal{F}^{-1} \left( \mathcal{F}(f(t)) \mathcal{F}(\Pi(t)) \right)$$

→ BUT WHAT DOES THIS DO IN FREQUENCY DOMAIN?!



DISADVANTAGES OF THE BOXCAR:

$$\mathcal{F}(\Pi(t)) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin(\omega/2)}{\omega} = 2\sqrt{\frac{2}{\pi}} \text{SINC}(\omega/2)$$

SO  $S(t) \propto \mathcal{F}^{-1}(\tilde{F}(\omega) \text{SINC}(\omega/2))$

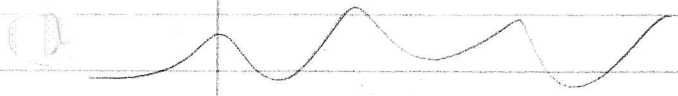
- RINGING
- FAT TAILS

ALTERNATIVE? GAUSSIAN SMOOTHING

$$S(t) = \int_{-\infty}^{\infty} f(t') e^{-(t-t')^2/2\sigma^2} dt'$$

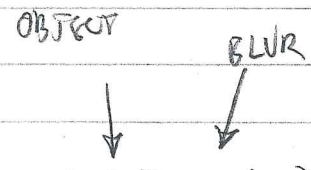
$$= \mathcal{F}^{-1}(\tilde{F}(\omega))$$

- NO RINGING IN k-SPACE
- TAILS FALL FAST



§13.1

DECONVOLUTION:  
("DEBLURRING")



SIGNAL  $S(t) = f(t) * p(t)$

$$\tilde{S}(\omega) = \tilde{F}(\omega) \tilde{P}(\omega) \sqrt{2\pi}$$

$$\tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\tilde{S}(\omega)}{\tilde{P}(\omega)}$$

SHOW HOW THE NOISE FLOOR MAKES THIS DIFFICULT!

$$\tilde{F}(\omega) \propto \frac{\tilde{S}(\omega)}{\tilde{P}(\omega)}$$

← NOISE FLOOR DOMINATES AT HIGH  $\omega$

← SMALL AT HIGH  $\omega$  (BY DEFINITION)

BUT HIGH  $\omega$  IS WHAT WE ARE TRYING TO RESTORE! THE INFORMATION MAY BE LOST.