

LAPLACE TRANSFORM

$$\mathcal{L}[f(t)] \equiv \bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt$$

LINEAR:

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

UNIQUE INVERSE: (SEE CH. 20 REGARDING \mathcal{L}^{-1}):

$$\mathcal{L}^{-1}[a\bar{f}(s) + b\bar{g}(s)] = af(t) + bg(t) \quad (13.56)$$

SEE TABLE 13.1

USEFUL PROPERTIES --- DERIVATIVES:

$$\begin{aligned} \mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^{\infty} \frac{df}{dt} e^{-st} dt \quad \text{BY PARTS... } \int u dv = uv - \int v du \\ &= \left[f(t) e^{-st} \right]_0^{\infty} + s \int_0^{\infty} f(t) e^{-st} dt \end{aligned}$$

$$\mathcal{L}\left[\frac{df}{dt}\right] = -f(0) + s\bar{f}(s), \quad s > 0 \quad (13.57)$$

(WHAT IS ASSUMED ABOUT $f(t)$?)

INTEGRALS:

MORE GENERALLY,

$$\mathcal{L}\frac{d^n f}{dt^n} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\begin{aligned} \mathcal{L}\left[\int_0^t f(u) du\right] &= \int_{t=0}^{\infty} e^{-st} \int_{u=0}^t f(u) du dt \quad \text{BY PARTS:} \\ &= \underbrace{\left[\frac{1}{s} e^{-st} \int_0^t f(u) du \right]_{t=0}^{\infty}}_0 + \underbrace{\int_0^{\infty} \frac{1}{s} e^{-st} f(t) dt}_{\frac{1}{s} \mathcal{L}[f(t)]} \end{aligned}$$

$$\mathcal{L}\left[\int_0^t f(u) du\right] = \frac{1}{s} \bar{f}(s) \quad (13.59)$$

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS
SAMPAD

TRANSLATION IN s AND t :

$$\mathcal{L}[e^{at} f(t)] = \int_0^{\infty} f(t) e^{at} e^{-st} dt = \int_0^{\infty} f(t) e^{-(s-a)t} dt$$

$$= \bar{f}(s-a) \quad (13.60)$$

$$\mathcal{L}[f(t-b)] = \int_b^{\infty} f(t-b) e^{-st} dt$$

$$\text{LET } \eta = t-b, \quad d\eta = dt$$

$$= \int_0^{\infty} f(\eta) e^{-s\eta} e^{-sb} d\eta$$

$$\mathcal{L}[f(t-b)] = e^{-sb} \bar{f}(s)$$

FIND & FIX THE ERROR.

OTHER PROPERTIES:

$$\mathcal{L}[f(at)] = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right) \quad (13.61)$$

$$\mathcal{L}[t f(t)] = -\frac{d}{ds} \bar{f}(s) \quad (\text{cf. } 13.62)$$

$$\mathcal{L}\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} \bar{f}(u) du \quad (13.63)$$

CONVOLUTION (SORT OF):

$$\mathcal{L}\left[\int_0^t f(u) g(t-u) du\right] = \bar{f}(s) \bar{g}(s) \quad (13.64)$$

↑
RHB (SLOPPILY) CALL THIS $f * g$, BUT
COMPARE (13.37)!

(1)

AND INVERTING
CALCULATING LAPLACE TRANSFORMS*

NOTE THAT THE \mathcal{L} INTEGRAL IS CARRIED OUT OVER $t > 0$, SO

$$\begin{aligned}\mathcal{L}(H(t)) &= \mathcal{L}(1) = \int_0^{\infty} e^{-st} dt \\ &= -\frac{1}{s} e^{-st} \Big|_{t=0}^{\infty}\end{aligned}$$

$$\mathcal{L}[1] = \frac{1}{s}.$$

CONVERGENCE: NOTE THAT s IS USUALLY TAKEN TO BE COMPLEX,

$$s = \sigma + i\omega \in \mathbb{C}; \quad \sigma, \omega \in \mathbb{R}$$

THE $t \rightarrow \infty$ LIMIT TAKEN ABOVE APPLIES ONLY IF

$$\text{Re}(s) = \sigma > 0.$$

MORE GENERALLY, THE "CONVERGENCE ABSCISSA" σ_0 IS DEFINED AS FOLLOWS:

GIVEN $f(t)$,

$\bar{f}(s)$ EXISTS FOR $\sigma > \sigma_0$.

* FOR THIS SECTION, I RELIED MAINLY ON CIRCUIT AND SYSTEM THEORY BY G.V. LAGO & L.M. BENNINGFIELD, (WILEY, 1979).

RECALL THAT

$$\mathcal{L}[e^{at}f(t)] = \bar{f}(s-a)$$

SO $\mathcal{L}[e^{at}] = \mathcal{L}[1](s-a) = \frac{1}{s-a}$.

THE CONVERGENCE ABCISSA IN THIS CASE IS $\sigma_c = a$.

IN GENERAL, THE REGION OF CONVERGENCE IS TO THE RIGHT OF THE RIGHT-MOST POLE IN THE S-PLANE.

TRIG FUNCTIONS:

$$\cosh bt \equiv \frac{e^{bt} + e^{-bt}}{2}$$

$$\mathcal{L}[\cosh bt] = \frac{1}{2} \left(\frac{1}{s-b} + \frac{1}{s+b} \right)$$

$$= \frac{1}{2} \left(\frac{s+b + s-b}{s^2 - b^2} \right)$$

$$\mathcal{L}[\cosh bt] = \frac{s}{s^2 - b^2}, \quad \sigma_c = +b.$$

SIMILARLY, SINCE $\cos(bt) = \cosh(ib t)$,

$$\mathcal{L}[\cos bt] = \frac{s}{s^2 + b^2}$$

QUESTION: WHAT IS THE CONVERGENCE ABCISSA FOR $\mathcal{L}[\cos bt]$?

WE HAVE $\frac{s}{s^2 + b^2} = \frac{s}{(s+ib)(s-ib)} \rightarrow$ POLES AT $s = \pm ib$

REAL PART IS $\sigma = 0$ FOR BOTH POLES, SO $\boxed{\sigma_c = 0}$.

NOW, SINCE $\mathcal{L}[tf(t)] = -\frac{d}{ds} F(s)$,

$$\mathcal{L}[t] = -\frac{d}{ds} \mathcal{L}[1] = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \frac{2}{s^3}$$

$$\mathcal{L}[t^3] = \frac{6}{s^4}$$

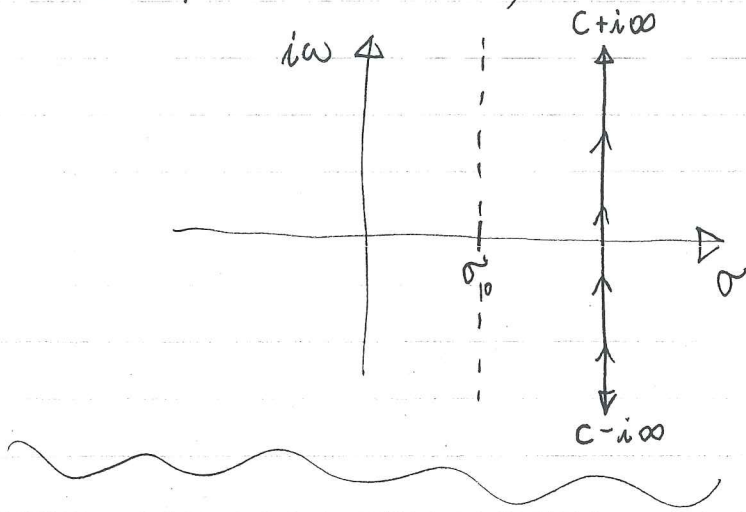
$$\boxed{\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}}$$

INVERSE TRANSFORMS

WHEN WE ENCOUNTER COMPLEX ANALYSIS, WE'LL SHOW THAT THE INVERSE LAPLACE TRANSFORM IS:

$$f(t) = \mathcal{L}^{-1}[\bar{f}(s)] = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \bar{f}(s) e^{st} ds$$

WHERE $C > \sigma_0$ (I.E. PATH OF INTEGRATION IS CHOSEN TO THE RIGHT OF ALL POLES)



BUT FOR NOW, WE'LL DEVELOP THE ART OF INVERTING LAPLACE TRANSFORMS BY INSPECTION.

ASIDE: PARTIAL FRACTIONS AND LAPLACE TRANSFORMS (SEE §1.4!) (5)

VERY OFTEN, WE'LL FIND FUNCTIONS LIKE

$$\bar{f}(s) = \frac{1}{(s+a)(s+b)}$$

AND FOR THIS, WE'LL NEED THE METHOD OF PARTIAL FRACTIONS:

$$\begin{aligned} \frac{1}{(s+a)(s+b)} &= \frac{K_a}{s+a} + \frac{K_b}{s+b} \\ &= \frac{K_a(s+b) + K_b(s+a)}{(s+a)(s+b)} \end{aligned}$$

$$\Rightarrow K_a(s+b) + K_b(s+a) = 1.$$

IF THIS IS TO HOLD $\forall s$, THEN $K_a = -K_b$.

$$\Rightarrow -K_b(s+b) + K_b(s+a) = 1$$

$$K_b(a-b) = 1$$

$$\therefore K_b = \frac{1}{a-b}, \quad K_a = \frac{1}{b-a}.$$

SO

$$\frac{1}{(s+a)(s+b)} = \frac{1}{(b-a)(s+a)} + \frac{1}{(a-b)(s+b)}.$$

THUS,

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{1}{(s+a)(s+b)} \right] &= \frac{1}{b-a} e^{-at} + \frac{1}{a-b} e^{-bt} \\ &= \left(\frac{1}{b-a} \right) (e^{-at} - e^{-bt}). \end{aligned}$$

PARTIAL FRACTIONS EXAMPLE WITH 3 POLES:

$$\frac{1}{(s+\alpha)(s+\beta)(s+\gamma)} = \frac{K_\alpha}{s+\alpha} + \frac{K_\beta}{s+\beta} + \frac{K_\gamma}{s+\gamma}$$

$$= \frac{K_\alpha(s+\beta)(s+\gamma) + K_\beta(s+\alpha)(s+\gamma) + K_\gamma(s+\alpha)(s+\beta)}{(s+\alpha)(s+\beta)(s+\gamma)}$$

ELIMINATE s^2 TERMS IN NUMERATOR:

$$K_\alpha + K_\beta + K_\gamma = 0$$

ELIMINATE s TERMS IN NUMERATOR:

$$K_\alpha(\beta+\gamma) + K_\beta(\alpha+\gamma) + K_\gamma(\alpha+\beta) = 0$$

CONSTANT TERMS IN NUMERATOR EQUAL UNITY:

$$K_\alpha\beta\gamma + K_\beta\alpha\gamma + K_\gamma\alpha\beta = 1.$$

THUS, WE HAVE 3 EQ'S AND 3 UNKNOWN'S $K_\alpha, K_\beta, K_\gamma$.

PARTIAL FRACTIONS WITH REPEATED ROOTS:

$$\text{E.G. } \frac{1}{(s+\alpha)(s+\beta)^n} = \frac{K_\alpha}{s+\alpha} + \frac{P_\beta(s)}{(s+\beta)^n}$$

$$\text{WHERE } P_\beta(s) = a_0 + a_1 s + \dots + a_{n-1} s^{n-1}$$

CONSIDER $n=2$:

$$\begin{aligned} \frac{1}{(s+\alpha)(s+\beta)^2} &= \frac{K_\alpha}{s+\alpha} + \frac{a+bs}{(s+\beta)^2} \\ &= \frac{K_\alpha(s+\beta)^2 + (a+bs)(s+\alpha)}{(s+\alpha)(s+\beta)^2} \end{aligned}$$

$$\text{SO } 1 = K_\alpha(s+\beta)^2 + (bs+a)(s+\alpha) \quad \forall s$$

$$s^2: K_\alpha + b = 0 \quad \Rightarrow \quad K_\alpha = -b$$

$$s: 2K_\alpha\beta + \alpha b + a = 0$$

$$1: K_\alpha\beta^2 + \alpha a = 1$$

$$\text{SOLUTION: } K_\alpha = -b = \frac{+1}{(\alpha-\beta)^2}, \quad a = \frac{\alpha-2\beta}{(\alpha-\beta)^2}$$

GENERAL RULE:

IN THE PARTIAL FRACTION EXPANSION, A REPEATED ROOT $(s+\beta)$ OF DEGREE n IS REPRESENTED BY A SINGLE TERM $P_\beta/(s+\beta)^n$ WHERE P_β IS A POLYNOMIAL OF DEGREE $n-1$.