

LAPLACE TRANSFORM SO FAR

$$\bullet \quad \mathcal{L}[f(t)] = \bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt, \quad s = \sigma + i\omega.$$

• PROPERTIES (FOR EXAMPLE):

$$\mathcal{L}\left[\frac{df}{dt}\right] = -f(0) + s\bar{f}(s), \quad s > 0$$

$$\mathcal{L}[e^{at} f(t)] = \bar{f}(s-a)$$

$$\mathcal{L}[f(t-b)] = e^{-sb} \bar{f}(s) \quad (\text{ASSUMING } f(t < 0) = 0!)$$

$$\mathcal{L}\left[\int_0^t f(u) g(t-u) du\right] = \bar{f}(s) \bar{g}(s) \quad (\text{LIKE CONVOLUTION THM})$$

• $\bar{f}(s)$ IS DEFINED FOR $\sigma > \sigma_0$, WHERE THE CONVERGENCE ABSCISSA σ_0 IS THE REAL PART OF THE RIGHT-MOST POLE OF $\bar{f}(s)$ IN THE $\sigma\omega$ -PLANE.

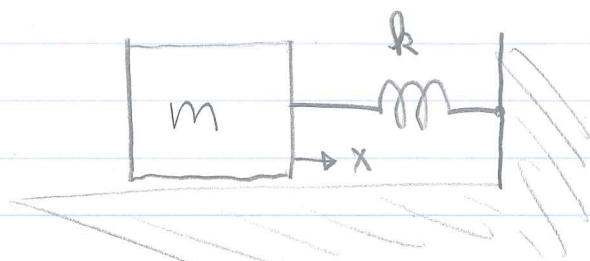
• SOME LAPLACE TRANSFORMS WERE CALCULATED. THE EXAMPLES IN TABLE 13.1 SHOULD BE STRAIGHTFORWARD FOR YOU TO CALCULATE.

• THE "TRICK" OF PARTIAL FRACTIONS WAS DESCRIBED FOR INVERTING LAPLACE TRANSFORMS.

TODAY'S QUESTION: WHAT IS IT GOOD FOR?

LAPLACE TRANSFORMS ARE COMMONLY USED TO SOLVE ODE INITIAL VALUE PROBLEMS. (§ 15.1.5)

EXAMPLE:



INITIAL COND'S:

$$x(0) = x_0$$

$$\dot{x}(0) = v_0$$

EOM FOR $x(t)$:

$$m\ddot{x} = -kx - \mu\dot{x} \leftarrow \text{"FRICTION"}$$

LAPLACE TRANSFORM THE EQUATION:

$$\mathcal{L}(\dot{x}) = -x(0) + s\bar{x}(s)$$

$$\mathcal{L}(\ddot{x}) = \mathcal{L}\left(\frac{d\dot{x}}{dt}\right) = -\dot{x}(0) + s\mathcal{L}(\dot{x})$$

$$= -\dot{x}(0) - s x(0) + s^2 \bar{x}(s)$$

ODE BECOMES:

$$-m v_0 - m s x_0 + m s^2 \bar{x}(s) = -k \bar{x} + \mu x_0 - m s \bar{x}(s)$$

SOLVE FOR $\bar{x}(s)$:

$$(m s^2 + \mu s + k) \bar{x} = (\mu + m s) x_0 + m v_0$$

$$\bar{x}(s) = \frac{(\mu + m s) x_0 + m v_0}{m s^2 + \mu s + k}$$

NOW WE HAVE A "SOLUTION" $\bar{x}(s)$. CAN WE INVERT FOR $x(t)$?

NOW SOLVE THESE SIMULTANEOUSLY:

$$K_- = x_0 - K_+, \text{ so}$$

$$K_+(a-b) + (x_0 - K_+)(a+b) = \frac{M}{m}x_0 + v_0$$

$$K_+ [(a-b) - (a+b)] = \left[\frac{M}{m} - (a+b) \right] x_0 + v_0$$

$$K_+ = - \frac{\left[\frac{M}{m} - (a+b) \right] x_0 + v_0}{2b}$$

$$K_- = \frac{\left[\frac{M}{m} - (a-b) \right] x_0 + v_0}{2b}$$

WE FIND CONSTANT SOLN'S FOR K_+ AND K_- , SO $\bar{x}(s)$ MAY BE EASILY INVERTED USING TABLE 13.1, WHICH WAS:

$$\mathcal{L}^{-1} \left(e^{\alpha t} \right) = \frac{1}{s - \alpha};$$

SINCE

$$\bar{x}(s) = \frac{K_+}{s+a+b} + \frac{K_-}{s+a-b}$$

$$\Rightarrow x(t) = K_+ e^{-(a+b)t} + K_- e^{-(a-b)t}$$

CONSTANTS INVOLVING
INITIAL CONDITIONS.

(5)

CHECK: DOES $x(t)$ SOLVE THE ODE?

$$x = e^{-(a \pm b)t} \quad (\text{TAKES INTO ACCOUNT FUNCTIONAL FORMS OF BOTH TERMS})$$

$$\dot{x} = -(a \pm b)x$$

$$\ddot{x} = (a \pm b)^2 x$$

PLUG INTO ODE, $m\ddot{x} + kx + \mu\dot{x} = 0$:

$$m(a \pm b)^2 + k - \mu(a \pm b) = 0 ?$$

$$m \left(\frac{\mu}{2m} \pm \frac{\sqrt{\mu^2 - 4mk}}{2m} \right)^2 + k - \mu \left(\frac{\mu}{2m} \pm \frac{\sqrt{\mu^2 - 4mk}}{2m} \right) = 0 ?$$

$$0 = \frac{\cancel{\mu}}{4m} + \frac{\mu \sqrt{\mu^2 - 4mk}}{2m} + \frac{\cancel{\mu^2 - 4mk}}{4m} + k - \frac{\cancel{\mu}}{2m} + \frac{\mu \sqrt{\mu^2 - 4mk}}{2m}$$

$$0 = 0 \quad \checkmark$$

(6)

CHECK: DOES $x(t)$ GET $x(0)$ AND $\dot{x}(0)$ RIGHT?

$$x(0) = K_+ + K_- \quad \checkmark \quad \leftarrow \text{AN EQUATION WE HAD USED TO GET } K_+ \text{ AND } K_-!$$

$$\dot{x}(t) = -K_+(a+b)e^{-(a+b)t} - K_-(a-b)e^{-(a-b)t}$$

$$\dot{x}(0) = -K_+(a+b) - K_-(a-b)$$

$$= \frac{(a+b) \left[\frac{M}{m} - (a+b) \right] x_0 + (a+b)v_0}{2b} - \frac{(a-b) \left[\frac{M}{m} - (a-b) \right] x_0 + (a-b)v_0}{2b}$$

$$= \frac{1}{2b} \left[2b \frac{M}{m} - (a+b)^2 + (a-b)^2 \right] x_0 + 2b v_0$$

$$= \left(\frac{M}{m} - \frac{4ab}{2b} \right) x_0 + v_0$$

$$= \left(\frac{M}{m} - 2a \right) x_0 + v_0 \quad \checkmark$$

IN REAL LIFE, NOBODY USES THE LAPLACE TRANSFORM TO SOLVE HOMOGENEOUS ODES. (WHY?)

THE LAPLACE TRANSFORM BECOMES MORE USEFUL WHEN THERE IS A DRIVING TERM:

$$\begin{array}{c}
 \downarrow \\
 m\ddot{x} + kx + \mu\dot{x} = mA(t)
 \end{array}$$

NOW, THE EQUATION TRANSFORMS TO:

$$\bar{x}(s) = \frac{(\mu + ms)x_0 + m v_0 + m\bar{A}(s)}{ms^2 + \mu s + k}$$

GETTING THE LAPLACE TRANSFORM OF THE ADDITIONAL TERM,

$$\frac{\bar{A}(s)}{s^2 + \frac{\mu}{m}s + \frac{k}{m}} \equiv \bar{g}(s)\bar{A}(s)$$

IS THEN REQUIRED TO GET THE RESULT. OR IS IT?

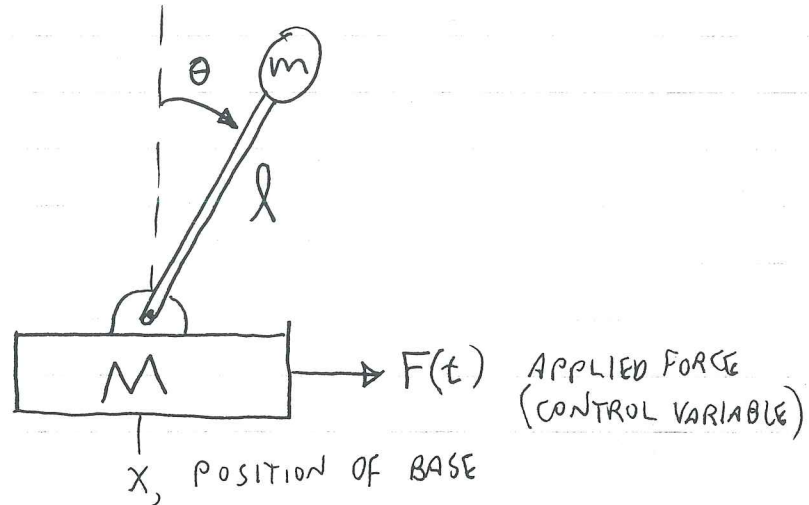
$$\begin{aligned}
 \mathcal{L}^{-1}(\bar{x}(s)) &= x_H(s) + \mathcal{L}^{-1}(\bar{g}(s)\bar{A}(s)) \\
 &\quad \uparrow \\
 &\quad \text{HOMOGENEOUS} \\
 &\quad \text{SOLN} \\
 &= x_H(s) + \int_0^t g(u) A(t-u) du
 \end{aligned}$$

(I USED THE \mathcal{L} VERSION OF THE CONVOLUTION THM)

LIMITATIONS: REQUIRE FORCING TERM $A(t) = 0$ FOR $t < 0$ (WHY?)

EXAMPLE: CONTROL THEORY FOR AN INVERTED PENDULUM

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ASSUME $M \gg m$, SO

$$M \ddot{x} = F$$

EQUATION OF MOTION FOR m : (ASSUME $\theta \ll 1$)

$$\text{ACCELERATION} = l \ddot{\theta} + \ddot{x}$$

$$\text{FORCE} = mg \sin \theta \rightarrow mg \theta$$

$$\underline{\text{SO}} \quad mg \theta = m(l \ddot{\theta} + \ddot{x})$$

$$\cancel{g \theta} = \cancel{l \ddot{\theta}} + \frac{F}{M}$$

$$l \ddot{\theta} - g \theta = -\ddot{x}(t)$$

HOW MAY WE CHOOSE $\ddot{x}(t)$ TO STABILIZE THE SYSTEM?

CLOSE THE "CONTROL LOOP" BY MAKING $\ddot{x}(t)$ DEPEND ON MEASURED STATE PARAMETERS.

CONTROL LAW #1

→ E.G. LET $\ddot{x} = a\theta$, a AN ADJUSTABLE PARAMETER.

$$\Rightarrow l\ddot{\theta} - (g-a)\theta = 0$$

$$\ddot{\theta} = \frac{g-a}{l}\theta$$

EVIDENTLY, $\theta = A e^{-\alpha t} + B e^{\alpha t}$

WHERE $\alpha = \sqrt{\frac{g-a}{l}}$

- IF $a \leq g$, THEN $\alpha \in \mathbb{R}$.

THIS IS UNSTABLE DUE TO THE GROWING EXPONENTIAL MODE.

- IF $a > g$, THEN $\alpha \in \mathbb{I}$

THIS IS OSCILLATING, STILL NOT WHAT WE WANT

CONCLUSION: $\ddot{x} = a\theta$ IS INSUFFICIENT FOR CONTROL.

CONTROL LAW

#2

→ NOW, TRY $\ddot{x}(t) = b\dot{\theta}$, SO

$$l\ddot{\theta} + b\dot{\theta} - g\theta = 0$$

ACTS LIKE A DAMPING TERM.

LET $\theta = e^{\alpha t}$,

$$l\alpha^2 + b\alpha - g = 0$$

$$\alpha = \frac{-b \pm \sqrt{b^2 + 4gl}}{2l}$$

CLEARLY, WE AGAIN HAVE A GROWING EXPONENTIAL MODE.

CONTROL LAW

#3

→ TRY $\ddot{x}(t) = a\theta + b\dot{\theta}$,

$$\text{SO } l\ddot{\theta} + b\dot{\theta} - (g-a)\theta = 0$$

AGAIN LET $\theta = e^{\alpha t}$,

$$l\alpha^2 + b\alpha - (g-a) = 0$$

$$\alpha = \frac{-b \pm \sqrt{b^2 + 4(g-a)l}}{2l}$$

FOR STABILITY, WANT $\text{Re}(\alpha) < 0$, SO REQUIRE

$$a \geq g$$

CAN ALSO PREVENT OSCILLATIONS BY CHOOSING:

$$b^2 + 4(g-a)l \geq 0 \Rightarrow a - g \leq b^2/4l$$

$$a \leq \frac{b^2}{4l} + g \quad \left[= \text{IMPLIES CRITICAL DAMPING} \right]$$

THAT'S ABOUT AS FAR AS WE CAN GO WITHOUT FOURIER OR LAPLACE TRANSFORMS... OR A GREEN'S FUNCTION!

IN "REAL LIFE", WE SHOULD BE STABILIZING THE PENDULUM AGAINST SOME EXTERNAL FORCING FUNCTION $G(t)$:

$$l\ddot{\theta} + b\dot{\theta} - (g-a)\theta = G(t)$$

ASIDE
FOR EXAMPLE, FORCE OF WIND RESISTANCE (DRAG) ON THE PENDULUM:

$$F_d = \frac{1}{2} \rho v^2 C_D$$

↑ AIR DENSITY ↑ $v(t) =$ WIND VELOCITY ← DRAG COEFFICIENT OF PENDULUM BOB

AND THEN $G(t) = \frac{C_D}{2m} \rho v^2(t)$

WE CAN ANALYZE THIS EQN. WITH EITHER FOURIER OR LAPLACE TRANSFORMS.

LAPLACE TRANSFORM APPROACH

TAKE LAPLACE TRANSFORM OF EQ. OF MOTION, USING

$$\mathcal{L} \left[\left(\frac{d}{dt} \right)^n f(t) \right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

THUS,

$$\mathcal{L}(\dot{\theta}) = s \bar{\theta} - \theta(0)$$

$$\mathcal{L}(\ddot{\theta}) = s^2 \bar{\theta} - s \theta(0) - \dot{\theta}(0)$$

EOM: $\mathcal{L} \ddot{\theta} + b \dot{\theta} - (g-a) \theta = G(t)$

$\mathcal{L}(\text{EOM}): \mathcal{L}(s^2 \bar{\theta} - s \theta(0) - \dot{\theta}(0)) + b(s \bar{\theta} - \theta(0)) - (g-a) \bar{\theta}(s) = \bar{G}(s).$

$$[s^2 + bs - (g-a)] \bar{\theta}(s) = \bar{G}(s) + (ls - b) \theta(0) + l \dot{\theta}(0)$$

$$\bar{\theta}(s) = \frac{\bar{G}(s) + (ls - b) \theta_0 + l \dot{\theta}_0}{s^2 + bs - (g-a)}$$

WE MAY SOLVE FOR $\theta(t)$ BY TAKING \mathcal{L}^{-1} .
 BUT FIRST, LET'S EXAMINE STABILITY...

WE KNOW THAT THE POLES (ROOTS OF THE DENOMINATOR) WILL EACH CORRESPOND TO AN EXPONENTIAL TERM IN THE SOLUTION $\theta(t)$.

EACH POLE IS LOCATED AT SOME $s = \sigma + i\omega$.

- $\omega \neq 0 \Rightarrow$ OSCILLATION
- $\sigma > 0 \Rightarrow$ EXPONENTIAL GROWTH (INSTABILITY)
- $\sigma < 0 \Rightarrow$ EXPONENTIAL DECAY (STABILITY)

AS BEFORE, THE PRESENCE OF EVEN ONE UNSTABLE MODE MEANS THE SYSTEM AS A WHOLE IS UNSTABLE.

FIND POLES: $\lambda s^2 + bs - (g-a) = 0$

$$s = \frac{-b \pm \sqrt{b^2 + 4\lambda(g-a)}}{2\lambda}$$

IT'S JUST THE SAME ^{AS THE} CHARACTERISTIC EQN. AS FOR THE HOMOGENEOUS ODE... CONCLUSIONS ARE THE SAME:

- ① $\sigma \leq 0$ IFF $a \geq g$
- ② $a \leq \frac{b^2}{4\lambda} + g$ GIVES US $\omega = 0$ (NO OSCILLATIONS)
- ③ LARGE $b \Rightarrow$ FAST DECAY.

- OUR EARLIER ANSATZ SOLN. GAVE US ALL THE STABILITY INFO MORE EASILY.

- ANSATZ SOLN. ALSO WOULD WORK FOR THE HOMOGENEOUS ($G(t)=0$) INITIAL VALUE PROBLEM.

+ LET'S LOOK AT HOW THE LAPLACE TRANSFORM HANDLES THE INHOMOGENEOUS INITIAL VALUE PROBLEM:

FOR SIMPLICITY, TAKE THE CRITICAL DAMPING CASE:

$$a = \frac{b^2}{4l} + g, \quad \text{SO THE POLES ARE AT}$$

$$s = \sigma = \frac{-b \pm 0}{2l}$$

AND WE MAY FACTOR $\bar{\Theta}(s)$ THIS WAY:

$$\bar{\Theta}(s) = \frac{\bar{G}(s)}{l \left(s + \frac{b}{2l} \right) \left(s + \frac{b}{2l} \right)}$$

FOR CONCRETENESS, USE A PULSED FORCE

$$G(t) = G_0 e^{-t/t_0}$$

$$\bar{G}(s) = \frac{G_0}{(s + 1/t_0)}$$

AND INITIAL CONDITIONS!

$$\Theta_0 = 0$$

$$\dot{\Theta}_0 = 0$$

$$\bar{\Theta} = \frac{G_0/l}{(s + 1/t_0) \left(s + \frac{b}{2l} \right)^2}$$

THE FORM OF $\bar{\Theta}(s)$ IS IDENTICAL TO OUR PARTIAL FRACTION EXAMPLE WITH REPEATED ROOTS...

$$\bar{\Theta}(s) = \frac{G_0/l}{\left(\frac{1}{t_0} - \frac{b}{2l}\right)^2} \left[\frac{1}{s + \frac{1}{t_0}} + \frac{\frac{1}{t_0} - \frac{b}{2l} - s}{(s + b/2l)^2} \right]$$

CONSIDER THE FOLLOWING TRANSFORM PAIRS,
ADAPTED FROM TABLE B.1:

$\frac{f(t)}{e^{-at}}$	$\frac{\bar{f}(s)}{1/(s+a)}$
$t e^{-at}$	$1/(s+a)^2$

REARRANGE $\bar{\Theta}(s)$: (VERY IMPORTANT TRICK!)

$$\bar{\Theta}(s) = \frac{G_0/l}{\left(\frac{1}{t_0} - \frac{b}{2l}\right)^2} \left[\frac{1}{\left(s + \frac{1}{t_0}\right)} - \frac{1}{\left(s + b/2l\right)} + \frac{\frac{1}{t_0} - \frac{b}{2l}}{\left(s + b/2l\right)^2} \right]$$

THUS,

$$\Theta(t) = \frac{G_0/l}{\left(\frac{1}{t_0} - \frac{b}{2l}\right)^2} \left[e^{-t/t_0} - e^{-bt/2l} + \left(\frac{1}{t_0} - \frac{b}{2l}\right) t e^{-bt/2l} \right].$$

FOURIER TRANSFORM APPROACH

EOM: $l\ddot{\theta} + b\dot{\theta} - (g-a)\theta = G(t)$

\mathcal{F} (EOM): $-l\omega^2 \tilde{\theta} + i\omega b \tilde{\theta} - (g-a)\tilde{\theta} = \tilde{G}(\omega)$

$$\tilde{\theta}(\omega) = \frac{\tilde{G}(\omega)}{-l\omega^2 + i\omega b - (g-a)}$$

ONE ADVANTAGE OF THE F.T. IS THAT WE GET A FREQUENCY RESPONSE — A GAIN (OR ATTENUATION) AND A PHASE LAG (OR LEAD) AS A FUNCTION OF ω .

THE FREQUENCY RESPONSE ALSO CONTAINS STABILITY INFORMATION. ROOTS OF THE DENOMINATOR:

$$\omega_0 = \frac{ib \pm \sqrt{-b^2 - 4l(g-a)}}{2l}$$

INTERPRETATION:

- $\text{Re}(\omega_0) \neq 0 \Rightarrow$ OSCILLATORY RESPONSE
- $\text{Im}(\omega_0) > 0 \Rightarrow$ STABLE (DECAYING) MODE
- $\text{Im}(\omega_0) < 0 \Rightarrow$ UNSTABLE (GROWING) MODE

SEE THIS BY PLUGGING INTO $e^{i\omega_0 t}$.

INVERSION OF THE FOURIER TRANSFORM CAN BE DONE BY BREAKING OUT POLES AS INDIVIDUAL TERMS WITH THE PARTIAL FRACTION TECHNIQUE (JUST LIKE LAPLACE TRANSFORMS).

CHOOSE CRITICAL DAMPING CASE AGAIN:

$$\omega_0 = \frac{ib \pm 0}{2l}$$

$$\tilde{\Theta}(l) = \frac{-\tilde{G}(\omega)/l}{(\omega - ib/2l)^2} = \frac{\tilde{G}(\omega)/l}{(b/2l + i\omega)^2}$$

CONSIDER THIS FOURIER TRANSFORM PAIR (FROM BRACEWELL):

$$\mathcal{F}(te^{-t}H(t)) = \frac{1}{\sqrt{2\pi}}(1+i\omega)^{-2}$$

RECALL
 $\mathcal{F}(f(\beta t)) = \frac{1}{\beta} \tilde{f}(\omega/\beta)$
 $\Rightarrow \tilde{f}(\frac{\omega}{\beta}) = \mathcal{F}(\beta f(\beta t))$

Now,

$$\frac{1}{(b/2l - i\omega)^2} = \frac{4l^2}{b^2} \left(1 - i\omega \frac{2l}{b}\right)^{-2}$$

$$= \frac{4l^2}{b^2} \sqrt{2\pi} \mathcal{F}\left[\frac{b}{2l} \left(\frac{bt}{2l}\right) e^{-bt/2l} H(t)\right]$$

$$= \sqrt{2\pi} \mathcal{F}\left[H(t) te^{-bt/2l}\right]$$

NOW IF WE FOURIER TRANSFORM $\tilde{\Theta} \rightarrow \Theta$ AND USE THE CONVOLUTION THEOREM:

$$\Theta(t) = \frac{G(t)}{l} * \underbrace{\left[H(t) te^{-bt/2l}\right]}_{\text{GREEN'S FUNCTION (IMPULSE RESPONSE)}}$$

I COULD HAVE OBTAINED THE SAME RESULT FROM THE LAPLACE TRANSFORM BY USING THE APPROPRIATE CONVOLUTION THEOREM.

HOW MIGHT WE LEARN THE BEHAVIOR OF \dot{x} ,
THE POSITION OF THE BASE?

(OUR BALANCING ACT WOULD LOOK UNIMPRESSIONING
IF WE HAVE TO TAKE OFF RUNNING TO KEEP THE
PENDULUM UP!)

RECALL THE CONTROL LAW: (#3, THE SUCCESSFUL ONE)

$$\ddot{x} = a\theta + b\dot{\theta}$$

LET'S FOURIER TRANSFORM THIS:

$$-\omega^2 \tilde{x} = a\tilde{\theta} + i\omega b \tilde{\theta}$$

$$\tilde{x} = \frac{-(a+i\omega b)}{\omega^2} \tilde{\theta}$$

IN THE FOURIER DOMAIN, EVERYTHING IS RELATED ALGEBRAICALLY.
WE MAY CONVENIENTLY SOLVE FOR \tilde{x} ; AND IN TURN,
USING OUR FREQUENCY RESPONSE, \tilde{x} IN TERMS OF \tilde{G} :

$$\tilde{x} = \frac{-(a+i\omega b)}{\omega^2} \frac{\tilde{G}/l}{(b/2l + i\omega)^2} \quad \leftarrow \begin{array}{l} \text{MORE GENERALLY,} \\ -\omega^2 + i b \frac{\omega}{l} - (g-a)/l \end{array}$$

RECALL THAT WE HAD CHOSEN $a = \frac{b^2}{4l} + g$ (CRITICAL DAMPING),

$$\tilde{x} = \frac{-(\frac{b^2}{4l} + g + i\omega b)}{\omega^2}$$

THE KEY THING HERE IS THAT WE HAVE AN ω NOW
IN THE DENOMINATOR. THERE IS EVIDENTLY A MODE
THAT LOOKS LIKE

$$x \sim Ae^{i\omega t} + Bte^{i\omega t}$$

↙ SINCE IT'S ω^2

IN OTHER WORDS, x IS JUST AS STABLE AS θ ,
EXCEPT THAT IT CAN COME TO REST ANYWHERE —
DISPLACED FROM ITS ORIGINAL POSITION.

BRAINSTORM: WOULD IT BE POSSIBLE TO
RETURN THE BASE (x) TO AN
EQUILIBRIUM POSITION? ($x=0$)

IDEA: CONTROL LAW MODIFICATION

$$\ddot{x} = a\theta + b\dot{\theta} - \alpha x - \beta \dot{x}$$