

LAST TIME, I SHOWED YOU HOW TO SOLVE A LINEAR, INHOMOGENEOUS ODE WITH CONSTANT COEFFICIENTS USING THE LAPLACE TRANSFORM:

$$\overset{\text{DIFF. OPERATOR}}{\mathcal{L}} x(t) = \underset{\text{FORCING}}{f(t)}$$

\swarrow UNKNOWN

TAKE THE TRANSFORM,

$$\mathcal{L}(\mathcal{L} x) = \bar{f}(s)$$

AND WE OBTAIN AN ALGEBRAIC EQUATION:

$$\bar{g}(s) \bar{x}(s) = \bar{f}(s)$$

$$\bar{x}(s) = \frac{\bar{f}(s)}{\bar{g}(s)} = \bar{G}(s) \bar{f}(s)$$

WHERE $\bar{G}(s) \equiv \frac{1}{\bar{g}(s)} = \mathcal{L}(\mathcal{L}^{-1} x(t))$

THE USUAL SOLUTION (SEE PROB. 2) IS TO TAKE \mathcal{L}^{-1} OF BOTH SIDES. IF INSTEAD WE USE THE CONVOLUTION THEOREM,

$$x(t) = \underset{\text{GREEN'S FXN.}}{G(t)} * f(t)$$

AND THE SAME ARGUMENT CAN BE USED WITH THE FOURIER TRANSFORM.

- ★ WHAT DID I LEAVE OUT OF THE MATH JUST NOW?
- INITIAL CONDITIONS FROM $\mathcal{L}(\mathcal{L} x)$
- THE CONVOLUTION THM FOR \mathcal{L} IS NOT EXACTLY CONVOLUTION.

* HOW WOULD THE ABOVE DIFFER FOR A FOURIER TRANSFORM APPROACH?

THIS HAS BEEN A KIND OF LEAD-IN FOR GREEN'S FUNCTIONS, RHB §15.2.5.

IN WHAT FOLLOWS, WE SET ASIDE THE INTEGRAL TRANSFORMS AND DERIVE THE GREEN'S FUNCTION APPROACH FROM ENTIRELY DIFFERENT CONSIDERATIONS.

INTRODUCTION TO GREEN'S FUNCTIONS — CONCEPTUAL...

SUPPOSE WE HAVE AN INHOMOGENEOUS, LINEAR ODE,

$$\mathcal{L} y(x) = f(x), \quad x \in \mathbb{D} \quad (1)$$

SUPPOSE FURTHER THAT WE HAVE A GENERAL SOLUTION TO THE COMPLEMENTARY (HOMOGENEOUS) EQUATION,

$$\mathcal{L} y_c = 0.$$

IT IS THEN ONLY A LITTLE HARDER TO SOLVE

$$\mathcal{L} G(x, z) = \delta(x-z), \quad z \in \mathbb{D}. \quad (2)$$

- SAME AS COMPLEMENTARY EQUATION WHEN $x < z$ OR $x > z$.
- REQUIRES CAREFUL MATCHING ACROSS $x = z$ BOUNDARY.

THEN IF WE INTEGRATE (2), TIMES OUR FORCING $f(z)$,

$$\int_{\mathbb{D}} dz f(z) \mathcal{L} G(x, z) = \int_{\mathbb{D}} dz f(z) \delta(x-z)$$

$$\Rightarrow \mathcal{L} \int_{\mathbb{D}} dz f(z) G(x, z) = f(x). \quad (\text{cf. (1)!})$$

THUS, THE SOLUTION TO (1) IS

$$y(x) = \int_{\mathbb{D}} dz f(z) G(x, z).$$

NOW LET'S DO AN EXAMPLE WITH HOMOGENEOUS BOUNDARY CONDITIONS...

EXAMPLE

$$y'' + \frac{1}{4}y = f(x), \quad (3)$$

$$\text{B.C.'s: } y(0) = y(\pi) = 0$$

HOMOGENEOUS SOLN:

$$y'' + \frac{1}{4}y = 0$$

$$y = A \sin \frac{x}{2} + B \cos \frac{x}{2}$$

GREEN'S FUNCTION:

$$\text{WRITE } f(x) = \int_0^\pi f(z) \delta(x-z) dz$$

$$y(x) = \int_0^\pi f(z) G(x,z) dz$$

DOES $y(x)$ SATISFY THE B.C.'S? (discuss --- it will if $G(x,z)$ DOES!)

(3) BECOMES:

$$\int_0^\pi f(z) \left[\frac{\partial^2 G}{\partial x^2} + \frac{1}{4}G \right] dz = \int_0^\pi f(z) \delta(x-z) dz$$

$$\text{OR... } \frac{\partial^2 G}{\partial x^2} + \frac{1}{4}G = \delta(x-z) \quad (4)$$

REGION 1: $x < z$

$$G_1(x,z) = A \sin \frac{x}{2} + B \cos \frac{x}{2}, \quad x < z$$

$$G_1(0,z) = 0 \Rightarrow B = 0.$$

$$G_1(x,z) = A \sin \frac{x}{2}$$

REGION 2: $x \geq z$

$$G_2(x,z) = C \sin \frac{x}{2} + D \cos \frac{x}{2}$$

$$G_2(\pi, z) = 0 \Rightarrow C = 0$$

$$G_2(x,z) = D \cos \frac{x}{2}$$

NOW, WHAT SHOULD HAPPEN AT $x = z$?

• NOW, WE WANT $G(x, z) = \begin{cases} G_1, & x < z \\ G_2, & x \geq z \end{cases}$ TO SATISFY EQUATION (4) IN THE NEIGHBORHOOD OF $x = z$.

• IT IS CLEAR THAT ONE OF THE LHS TERMS MUST CONTAIN A $\delta(x-z)$. IT MUST BE THE HIGHEST DERIVATIVE, IF IT WERE NOT THE HIGHEST DERIVATIVE TERM, THEN THE DERIVATIVES HIGHER THAN THE ONE CONTAINING $\delta(x-z)$ WOULD CONTAIN DERIVATIVES OF $\delta(x-z)$, AND IT WOULD BE IMPOSSIBLE TO SATISFY THE ODE.

• SINCE THERE IS A $\delta(x-z)$ IN $\frac{\partial^2 G}{\partial x^2}$, THERE MUST ALSO BE A DISCONTINUITY IN THE NEXT LOWER DERIVATIVE, $\partial G / \partial x$. LOWER DERIVATIVES ARE CONTINUOUS! LET'S INTEGRATE (4) OVER AN INFINITESIMAL NEIGHBORHOOD CENTERED ON $x = z$:

$$\int_{z-\epsilon}^{z+\epsilon} dx \left[\frac{\partial^2 G}{\partial x^2} + \frac{1}{4} G \right] = \int_{z-\epsilon}^{z+\epsilon} dx \delta(x-z), \quad \epsilon \rightarrow 0.$$

↑
THIS TERM CONTAINS $\delta(x-z)$.

↑
THIS TERM DOES NOT CONTRIBUTE.

$$\therefore \frac{\partial G}{\partial x} \Big|_{x=z+\epsilon} - \frac{\partial G}{\partial x} \Big|_{x=z-\epsilon} = 1, \quad \text{OR ---}$$

$$\frac{\partial G_2}{\partial x} \Big|_{x=z} - \frac{\partial G_1}{\partial x} \Big|_{x=z} = 1.$$

$$-\frac{1}{2} D \sin \frac{z}{2} - \frac{1}{2} A \cos \frac{z}{2} = 1 \quad (5)$$

WE INSIST UPON CONTINUITY FOR LOWER ORDER TERMS. IN THIS CASE, ONLY G REMAINS: ^{AT $x=z$}

$$G_1(z, z) = G_2(z, z)$$

$$\therefore A \sin \frac{z}{2} = D \cos \frac{z}{2} \quad (6)$$

~~$k\epsilon = D = A$~~

COMBINING (5) & (6), WE SOLVE FOR A & D:

$$-\frac{1}{2}D \sin \frac{z}{2} - \frac{1}{2}D \cot \frac{z}{2} \cos \frac{z}{2} = 1$$

$$-\frac{1}{2}D \left\{ \sin^2 \frac{z}{2} + \cos^2 \frac{z}{2} \right\} = \sin \frac{z}{2}$$

$$D = -2 \sin \frac{z}{2}$$

$$A = -2 \cos \frac{z}{2}$$

THE GREEN'S FUNCTION IS THUS

$$G(x, z) = \begin{cases} -2 \cos \frac{z}{2} \sin \frac{x}{2}, & x < z \\ -2 \sin \frac{z}{2} \cos \frac{x}{2}, & x \geq z \end{cases}$$

AND THE SOLUTION TO THE INHOMOGENEOUS ODE (3) IS

$$\begin{aligned} y(x) &= \int_0^{\pi} f(z) G(x, z) dz \\ &= \underbrace{-2 \cos \frac{x}{2} \int_0^x f(z) \sin \frac{z}{2} dz}_{x \geq z, \text{ USE } G_2} - \underbrace{2 \sin \frac{x}{2} \int_x^{\pi} f(z) \cos \frac{z}{2} dz}_{x < z, \text{ USE } G_1} \end{aligned}$$