

PROCEDURE FOR USING GREEN'S FUNCTIONS TO SOLVE AN INHOMOGENEOUS, LINEAR ODE SUBJECT TO HOMOGENEOUS BOUNDARY CONDITIONS:

I. EQUATION OF MOTION

$$\mathcal{D}y = \sum_{m=0}^n \frac{a_m(x)}{\substack{\text{POSSIBLY NON-} \\ \text{CONST. COEFF'S}}} \frac{\partial^m y}{\partial x^m} = f(x), \quad a < x < b$$

II. SOLVE THE HOMOGENEOUS PROBLEM, $\mathcal{D}y_c = 0$

III. CONSTRUCT THE GREEN'S FUNCTION $G(x, z)$ FROM THE GENERAL SOLUTION $y_c(x)$. APPLY THE FOLLOWING BOUNDARY CONDITIONS:

a. LET $G(x, z)$ SATISFY THE SAME BOUNDARY CONDITIONS AS THE EOM, E.G. $y=0$ @ $x=a, b$

b. AT $x=z$, $a_n(x) \left(\frac{\partial}{\partial x} \right)^{n-1} G(x, z) \Big|_{x=z-\epsilon}^{z+\epsilon} = 1, \quad a < z < b.$

$$a_m(x) \left(\frac{\partial}{\partial x} \right)^m G(x, z) \Big|_{x=z-\epsilon}^{z+\epsilon} = 0, \quad m < n-1$$

NOW, WRITE $G(x, z) = \begin{cases} G_1(x, z), & x < z \\ G_2(x, z), & x > z \end{cases}$

THE ABOVE CONDITIONS (b) BECOME

$$\left(\frac{\partial}{\partial x} \right)^{n-1} (G_2 - G_1) \Big|_{x=z} = \frac{1}{a_n(z)}$$

AND $\left(\frac{\partial}{\partial x} \right)^m (G_2 - G_1) \Big|_{x=z} = 0, \quad m < n-1$

IV. $y(x) = \int_a^b f(z) G(x, z) dz$

$$y(x) = \int_a^x f(z) G_2(x, z) dz + \int_x^b f(z) G_1(x, z) dz \quad \star$$

$x > z$, USE G_2 $x < z$, USE G_1

★ JUSTIFICATION (FROM P.4):

$G(x, z)$ IS TO SATISFY:

$$\underbrace{\sum_{m=0}^n a_m(x) \frac{\partial^m}{\partial x^m} G(x, z)}_{\mathcal{L}G} = \delta(x-z)$$

INTEGRATE THIS EQUATION: ($\lim z \rightarrow 0$ IS IMPLIED HERE)

$$\int_{z-\epsilon}^{z+\epsilon} \sum_{m=0}^n a_m(x) \frac{\partial^m}{\partial x^m} G(x, z) dx = 1$$

ASSUME ONLY THE HIGHEST DERIVATIVE CONTRIBUTES!

$$\int_{z-\epsilon}^{z+\epsilon} \underbrace{a_n(x)}_u \underbrace{\left(\frac{\partial^n}{\partial x^n} G\right)}_{dv} dx = 1$$

CONTAINS $\delta(x-z)$

BY PARTS:

$$\underbrace{a_n(x)}_u \underbrace{\left(\frac{\partial^{n-1}}{\partial x^{n-1}} G\right)}_v \Big|_{z-\epsilon}^{z+\epsilon} - \int_{z-\epsilon}^{z+\epsilon} \underbrace{\left(\frac{\partial^{n-1}}{\partial x^{n-1}} G\right)}_v \underbrace{a_n'}_{du} dx = 1$$

CONTAINS $H(x-z)$

○ (WHY?)

FINITE INTEGRAND OVER INFINITESIMAL LIMITS

QED

Q: → WHAT WAS ASSUMED ABOUT THE COEFFICIENTS $a_m(x)$?

A: THEY DON'T CONTAIN δ FUNCTIONS, I.E. THEY ARE ~~NOT~~ CONTINUOUS.

* MORE GENERALLY, HOMOGENEOUS BOUNDARY CONDITIONS INCLUDE:

① DIRICHLET

$$y(a) = y(b) = 0$$

② NEUMANN

$$y'(a) = y'(b) = 0$$

③ MIXED (NEUMANN & DIRICHLET)

$$y(a) = 0 \text{ \& } y'(b) = 0$$

④ PERIODIC

$$y(a) - y(b) = 0 \text{ \& }$$

$$y'(a) - y'(b) = 0$$

EXAMPLES SHOWN
ARE FOR A
2nd - ORDER ODE.

THE NICE THING ABOUT HOMOGENEOUS BC'S IS THAT ANY LINEAR COMBINATION OF SOLUTIONS SATISFYING THEM WILL ALSO SATISFY THEM.

WHAT IF THE B.C.'S ARE INHOMOGENEOUS?

→ CAUSES PROBLEMS WITH SATISFYING
B.C.'S WHEN SUPERIMPOSING
GREEN'S FUNCTIONS TO CONSTRUCT SOLN.

CONNECTION TO LAPLACE TRANSFORMS

OFTEN, IT TURNS OUT THAT

$$G(x, z) = g(x-z). \quad (\text{TRANSLATIONALLY INVARIANT PROBLEM})$$

IN THAT CASE, WE HAVE \star BECOMING

$$y(x) = \int_a^x f(z) g_2(x-z) dz + \int_x^b f(x) g_1(x-z) dz \quad \star \star$$

NOW RECALL FROM LAPLACE TRANSFORM THEORY:

$$\mathcal{L} \left[\int_0^x f(u) g(x-u) du \right] = \bar{f}(s) \bar{g}(s)$$

$$\text{OR } \int_0^x f(u) g(x-u) du = \mathcal{L}^{-1} \left[\bar{f}(s) \bar{g}(s) \right]$$

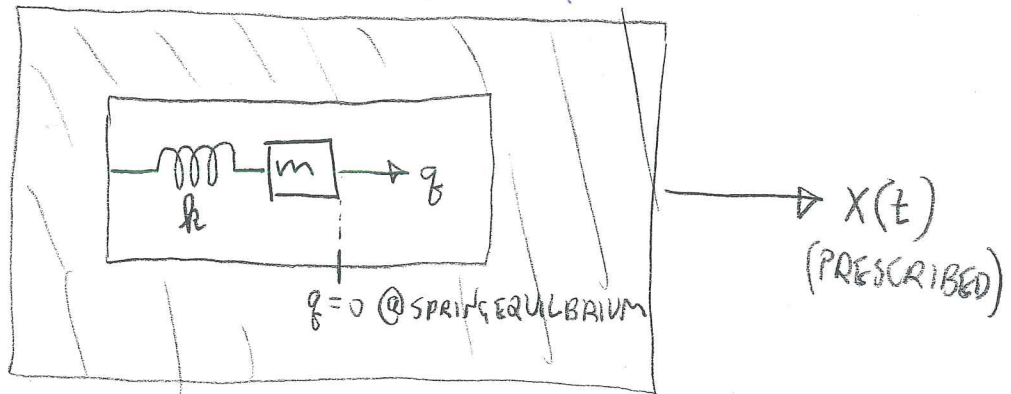
IF WE CHANGE VARIABLES IN $\star \star$ TO GET THE LIMITS RIGHT, THEN THE INTEGRAL WILL BE IN A FORM THAT CAN BE SOLVED USING TABLES OF LAPLACE TRANSFORMS.

SKIP.

IN-CLASS EXERCISE

— PASSENGER IN A CAR WITH SPRING SEATS!

ASSUME
AT REST
@ $t=0$



MASS & SPRING EMBEDDED IN A MOVING OBJECT

I. FIND EQ. OF MOTION / PUT IN FORM $D^2 q = f(t)$

$$m \frac{d^2}{dt^2} (q+x) = -kq$$

ACCEL. OF M IN LAB FRAME

$$m \ddot{q} + kq = -m \ddot{x}(t) \quad (1)$$

~~PAGE~~
~~MICHAEL~~

II. WHAT'S THE COMPLEMENTARY EQN?

$$m \ddot{q}_c + kq_c = 0$$

FIND GENERAL SOLN'S...

$$q_c = A e^{\alpha t}$$

$$m \alpha^2 + k = 0$$

$$\alpha = \pm \sqrt{\frac{k}{m}}$$

$$= \pm i \sqrt{\frac{k}{m}} \quad \dots \text{LET } \underline{\underline{\omega_0 = \sqrt{\frac{k}{m}}}}$$

$$q_c = A e^{i\omega_0 t} + B e^{-i\omega_0 t}$$

III. SUPPOSE SYSTEM @ $t = 0$ AT REST. FIND GREEN'S FUNCTION.

$$G(t, t_0) = \begin{cases} G_1, & t < t_0 \\ G_2, & t \geq t_0 \end{cases}$$

$$G_1 = Ae^{i\omega t} + Be^{-i\omega t}$$

$$\text{B.C.: } G_1(0, t_0) = \dot{G}_1(0, t_0) = 0$$

$$\Rightarrow A = B = 0 \Rightarrow G_1 = 0$$

$$G_2 = Ce^{i\omega t} + De^{-i\omega t}$$

$$m\ddot{G} + kG = \delta(t - t_0) \quad (2)$$

INTEGRATE:

$$\lim_{\epsilon \rightarrow 0} \dot{G} \Big|_{t=t_0-\epsilon}^{t_0+\epsilon} = 1/m$$

$$\dot{G}_2(t_0, t_0) - \dot{G}_1(t_0, t_0) = 1/m$$

$$\Rightarrow i\omega [C e^{i\omega t_0} - D e^{-i\omega t_0}] = \frac{1}{m} \quad (3)$$

CONTINUITY OF G :

$$G_1(t_0, t_0) = G_2(t_0, t_0)$$

$$C e^{i\omega t_0} + D e^{-i\omega t_0} = 0$$

$$\Rightarrow D = -C e^{2i\omega t_0} \quad (4)$$

PLUG (4) INTO (3):

$$C e^{i\omega t_0} + C e^{i\omega t_0} = \frac{-i}{m\omega}$$

$$C = \frac{-i}{2m\omega} e^{-i\omega t_0}$$

$$D = \frac{i}{2m\omega} e^{i\omega t_0}$$

CAREFUL



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$$G_2(t, t_0) = \frac{-i}{2m\omega} e^{-i\omega t} e^{i\omega t} + \frac{i}{2m\omega} e^{i\omega t} e^{-i\omega t}$$

$$= \frac{i}{2m\omega} \left[e^{-i\omega(t-t_0)} - e^{i\omega(t-t_0)} \right]$$

$$= \frac{i}{2m\omega} \left[(\cos \omega(t-t_0) - i \sin \omega(t-t_0)) - (\cos \omega(t-t_0) + i \sin \omega(t-t_0)) \right]$$

$$= \frac{i}{2m\omega} \left[-2i \sin \omega(t-t_0) \right]$$

$$G_2(t, t_0) = \frac{1}{m\omega} \sin \omega(t-t_0)$$

← E.R. THIS WAS GUARANTEED TO HAPPEN. WHY? WHAT DOES THE RESULT MEAN?

THUS, $G(t, t_0) = \begin{cases} 0, & t < t_0 \\ \frac{1}{m\omega} \sin \omega(t-t_0), & t \geq t_0 \end{cases}$

- WHAT ARE THE UNITS OF G? POSITION PER MOMENTUM
(AMPLITUDE IN METERS PER UNIT IMPULSE)
- HOW DOES THIS CONNECT TO ITS PHYSICAL INTERPRETATION?
- $G(t, t_0)$ IS THE RESPONSE OF THE SYSTEM TO AN IMPULSE, AN IMPULSE OF WHAT?

$$I = \int m \ddot{g} dt = \int F dt = \int \frac{dP}{dt} dt = \Delta P$$

IV. LET $\ddot{x}(t) = \begin{cases} a e^{-kt}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

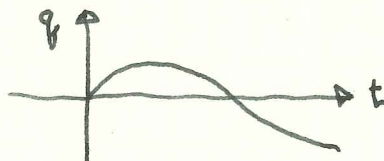
FIND $q(t)$, $t > 0$, AND SKETCH.

$$\begin{aligned} q(t) &= \int_0^{\infty} G(t, t_0) (-m \ddot{x}(t_0)) dt_0 \\ &= \int_0^t G_2(t, t_0) (-ma e^{-kt_0}) dt_0 \\ &\quad + \int_t^{\infty} \underbrace{G_1}_{0} (-m \ddot{x}(t_0)) dt_0 \\ &= \int_0^t -\frac{a}{\omega} e^{-kt_0} \sin \omega(t-t_0) dt_0 \\ &= -\frac{a}{\omega} \frac{K\omega}{(K^2 + \omega^2)} \left[\frac{1}{\omega} \sin \omega(t-t_0) + \frac{1}{K} \cos \omega(t-t_0) \right] e^{-kt_0} \Big|_{t_0=0}^t \\ &= \frac{-a\omega}{K^2 + \omega^2} \left[-K \sin \omega t + e^{-kt} \omega - \omega \cos \omega t \right] \\ &= \frac{a\omega}{K^2 + \omega^2} \left[K \sin \omega t + \omega \cos \omega t - \omega e^{-kt} \right] \end{aligned}$$

2 CASES:

(A) FOR $K \gg \omega$, THIS LOOKS LIKE THE GREEN'S FXN $G(t, 0)$:

$$q = \frac{a}{\omega K} \sin \omega t \quad (t \geq 0)$$



(B) FOR $K \ll \omega$, WE FIND THE RESPONSE TO A STEP FUNCTION FORCE:

$$q = \frac{a}{\omega^2} [\cos \omega t - e^{-kt}] \quad \text{IN } \lim_{K \rightarrow 0} q = \frac{a}{\omega^2} [\cos \omega t - 1]$$

