

NON-CONSTANT COEFF'S EXAMPLE

## "ELECTROSTATICS IN SPHERICAL SYMMETRY"

$$\begin{aligned}\nabla \cdot \vec{E} &= \rho / \epsilon_0 \\ \nabla \times \vec{E} &= 0\end{aligned}$$

SOLUTION IN TERMS OF SCALAR POTENTIAL,

$$\vec{E} = -\nabla \Phi \quad (\nabla \times \vec{E} = 0 \text{ AUTOMATICALLY})$$

$$\nabla^2 \Phi = -\rho / \epsilon_0$$

IN SPHERICAL SYMMETRY,  $\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \theta} = 0$ .

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) = -\rho(r) / \epsilon_0$$

THIS IS AN ODE:

$$\frac{d^2 \Phi}{dr^2} + \frac{2}{r} \frac{d\Phi}{dr} = -\frac{\rho(r)}{\epsilon_0}$$

GREEN'S FXN:

$$\frac{d^2}{dr^2} G(r, r') + \frac{2}{r} \frac{d}{dr} G(r, r') = -\frac{\delta(r-r')}{\epsilon_0} \quad \star$$

$$\left( \text{SOLN. WILL BE } \Phi(r) = \int_0^\infty \rho(r') G(r, r') dr' \right)$$

TRY TO CONFUSE EVERYBODY: WHY ISN'T IT:

$$\Phi(r) = \int_0^\infty -\frac{\rho(r')}{\epsilon_0} G(r, r') dr'$$

ANSWER: G IS RESPONSE  
TO  $\rho(r) = \delta(r-r')$   
SO IF WE USE  $\rho(r) = \delta(r'-a)$ ,  
WE SHOULD GET  $\Phi = G(r, a)$

COMPLEMENTARY EQN:

$$\frac{d^2}{dr^2} \Phi_c + \frac{2}{r} \frac{d}{dr} \Phi_c = 0$$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi_c}{dr} \right) = 0$$

THIS MEANS  $r^2 \frac{d\Phi_c}{dr} = A$  (CONSTANT.)

$$\int d\Phi_c = \int A \frac{dr}{r^2}$$

$$\Phi_c = \frac{-A}{r} + B$$

INTEGRATION  
CONST.

WE THEREFORE HAVE

$$G(r, r') = \begin{cases} \frac{-A}{r} + B, & r < r' \\ \frac{-C}{r} + D, & r > r' \end{cases}$$

BOUNDARY CONDITIONS — WHAT WOULD BE APPROPRIATE?

$$\Phi \rightarrow 0 \text{ AS } r \rightarrow \infty, \text{ SO } \underline{D = 0.}$$

A MORE SUBTLE POINT:  $\vec{E} = 0$  AT  $r = 0$  (SINCE ANY FINITE VECTOR  $\vec{E}$  WOULD VIOLATE SPHERICAL SYMMETRY AT THE ORIGIN):  $\frac{d\Phi}{dr} \rightarrow 0$  AT  $r = 0$ , SO  $\underline{A = 0.}$

(SO B.C.'S ARE HOMOGENEOUS, MIXED NEUMANN-DIRICHLET)

INTEGRATING  $\Delta$  ACROSS THE  $r=r'$  BOUNDARY,

$$\left. \frac{dG}{dr} \right|_{r'+\epsilon} = \frac{-1}{\epsilon_0} \left. \frac{dG}{dr} \right|_{r'-\epsilon}$$

$$\text{Now, } \frac{dG}{dr} = \begin{cases} 0, & r < r' \\ C/r^2, & r > r' \end{cases}$$

SO AT  $r=r'$ ,

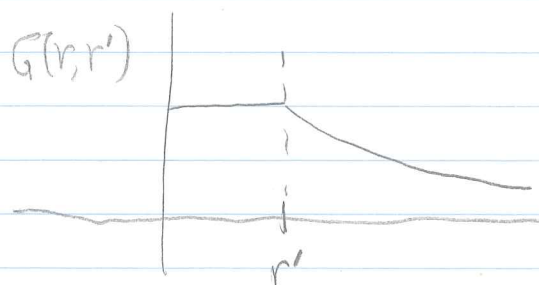
$$\frac{C}{r'^2} = \frac{-1}{\epsilon_0}$$

$$C = -\frac{r'^2}{\epsilon_0}$$

WE REQUIRE CONTINUITY OF  $G$  AT  $r=r'$ , SO

$$B = \frac{-C}{r'} = \frac{r'}{\epsilon_0}$$

$$G(r, r') = \begin{cases} \frac{r'}{\epsilon_0}, & r < r' \\ \frac{r'^2}{\epsilon_0 r}, & r > r' \end{cases}$$



} DOES THIS MAKE SENSE?  
(PHYSICALLY)

$$\Phi = \int_0^{\infty} G(r, r') \rho(r') dr'$$

$$\Phi(r) = \int_{r'=0}^r \frac{r'^2}{\epsilon_0 r} \rho(r') dr' + \int_{r'=r}^{\infty} \frac{r'}{\epsilon_0} \rho(r') dr'$$

IS IT RIGHT? CHECK!

LET  $\rho(r) = \begin{cases} \frac{q}{\frac{4}{3}\pi a^3}, & r < a \\ 0, & r > a \end{cases}$  } Q: PHYSICALLY, WHAT IS THIS?

FOR  $r > a$ ,

$$\begin{aligned} \Phi(r) &= \frac{q}{\frac{4}{3}\pi a^3} \int_{r'=0}^a \frac{r'^2}{\epsilon_0 r} dr' \\ &= \left( \frac{q}{\frac{4}{3}\pi a^3} \right) \frac{1}{\epsilon_0 r} \frac{a^3}{3} = \frac{q}{4\pi\epsilon_0 r} \quad \checkmark \end{aligned}$$

FOR  $r < a$ ,

$$\begin{aligned} \Phi(r) &= \frac{q}{\frac{4}{3}\pi a^3} \left[ \int_{r'=0}^r \frac{r'^2}{\epsilon_0 r} dr' + \int_{r'=r}^a \frac{r'}{\epsilon_0} dr' \right] \\ &= \frac{q}{\frac{4}{3}\pi a^3} \left[ \frac{r^2}{3\epsilon_0} + \frac{a^2}{2\epsilon_0} - \frac{r^2}{2\epsilon_0} \right] \\ &= \frac{q}{4\pi\epsilon_0} \left[ \frac{3}{2a} - \frac{r^2}{2a^3} \right] \quad \checkmark \end{aligned}$$

EXAMPLE: GREEN'S FUNCTION FOR A  
FIRST-ORDER ODE

①

$$\nabla \cdot \vec{E} = \rho / \epsilon_0 \quad \text{IN SPHERICAL COORDINATES,}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \theta} = 0, \quad \text{SYMMETRIC ABOUT ORIGIN}$$

$$\text{SO, } \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E) = \frac{\rho}{\epsilon_0}$$

$$\boxed{\frac{dE}{dr} + \frac{2}{r} E = \frac{\rho}{\epsilon_0}}$$

BOUNDARY CONDITION:  $E(0) = 0$

ONLY 1 B.C.  
SINCE THE ODE  
IS FIRST ORDER!

COMPLEMENTARY EQUATION:

$$\frac{dE_c}{dr} + \frac{2}{r} E_c = 0$$

$$\frac{dE_c}{E_c} = -2 \frac{dr}{r} \Rightarrow E_c = \frac{A}{r^2}$$

GREEN'S FUNCTION IS THEREFORE OF THE FORM:

$$G(r, r') = \begin{cases} \frac{A}{r^2}, & r < r' \\ \frac{B}{r^2}, & r > r' \end{cases}$$

BOUNDARY CONDITION  $\Rightarrow \underline{\underline{A=0}}$ .

$$\frac{dG}{dr} + \frac{2}{r} G = \delta(r-r') \quad *$$

$$\lim_{\epsilon \rightarrow 0} G \Big|_{r=r'-\epsilon}^{r'+\epsilon} = 1 \Rightarrow \frac{B}{r'^2} = 1$$

$$\therefore G(r, r') = \begin{cases} 0, & r < r' \\ \frac{r'^2}{r^2}, & r > r' \end{cases}$$

$$\therefore E(r) = \int_0^\infty G(r, r') \frac{\rho(r')}{\epsilon_0} dr' \quad **$$

$$= \int_0^r \frac{r'^2}{r^2} \frac{\rho(r')}{\epsilon_0} dr' + \int_r^\infty 0 dr'$$

$$E(r) = \frac{1}{\epsilon_0 r^2} \int_0^r \rho(r') r'^2 dr'$$

SAME ~~RESULT~~ RESULT AS GAUSS'S LAW:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{1}{\epsilon_0} \int_0^r \rho(r') \cdot \overbrace{4\pi r'^2 dr'}^{dV} \quad \checkmark$$



YOU MIGHT ASK WHY WE DID NOT SET UP OUR GREEN'S FUNCTION INTEGRAL (\*\*\*) WITH A GEOMETRICAL WEIGHTING FUNCTION:

$$E(r) = \int_0^\infty G(r, r') \frac{\rho(r')}{\epsilon_0} \textcircled{r'^2} dr'$$

IF WE DO THAT, THEN THE INTEGRATION OF  $\Delta G$  (\*) ACROSS THE BOUNDARY MUST BE DONE THE SAME WAY:

$$\int_{r=r'-\epsilon}^{r'+\epsilon} \left( \frac{dG}{dr} + \frac{2}{r} G \right) \textcircled{r^2} dr = 1 \Rightarrow \underline{\underline{B=1}}$$

SO  $E(r)$  COMES OUT THE SAME!

... AND LIKEWISE HAD I DONE  $\int dV \rightarrow \int 4\pi r^2 dr^2$ ,

SO LONG AS I HANDLE BOTH \* AND \*\* THE SAME WAY.

## INHOMOGENEOUS BOUNDARY CONDITIONS

5

SUPPOSE  $\mathcal{L}$  IS A 2<sup>nd</sup> ORDER, LINEAR DIFFERENTIAL OPERATOR.  
SOLVE FOR  $y(x)$ ,  $x > 0$ :

$$\boxed{\begin{aligned} \mathcal{L} y(x) &= f(x), & \text{SUBJECT TO} \\ y(0) &= A, & y'(0) = B \end{aligned}} \quad \star$$

- ① SOLVE  $\mathcal{L} y_c(x) = 0$  IN GENERAL. USE GENERAL SOLUTION  $y_c(x)$  TO CONSTRUCT A ~~SOME~~ PARTICULAR SOLUTION

$$y_1(x) \ni y_1(0) = A, \quad y_1'(0) = B.$$

- ② NOW SOLVE THE INHOMOGENEOUS EQUATION SUBJECT TO HOMOGENEOUS B.C.'S THAT CORRESPOND TO THE INHOMOGENEOUS ONES ABOVE:

$$\mathcal{L} y_2(x) = f(x), \quad y_2(0) = 0, \quad y_2'(0) = 0.$$

GREEN'S FUNCTION SOLUTION BY OUR USUAL PROCEDURE:

$$\text{IF } \mathcal{L} G(x, z) = \delta(x-z), \quad G(0, z) = 0, \quad \left. \frac{\partial G}{\partial x} \right|_{x=0} = 0,$$

$$\text{THEN } y_2(x) = \int_0^{\infty} G(x, z) f(z) dz.$$

[OF COURSE  $G$  IS BUILT FROM  $y_c$  AS USUAL]

- ③ TO SOLVE THE ORIGINAL PROBLEM  $\star$ ,

$$y(x) = y_1 + y_2. \quad \text{THAT'S ALL!}$$

→ DISCUSS - IS IT CORRECT?

- CAN YOU GENERALIZE TO DIFFERENT KINDS OF LINEAR OPERATORS & B.C.'S?

→ THE HOMEWORK EXPLORES A DIFFERENT APPROACH SET FORTH IN THE TEXT.