

LAST TIME §§ 17.1 - 17.3

EIGENFUNCTIONS AND DIRAC NOTATION, E.G.

$\mathcal{D}|y_i\rangle = \lambda_i|y_i\rangle$, $\langle f|g\rangle = \int_a^b f^*(x)g(x)p(x)dx$ **

HERMITIAN OPERATORS ($\mathcal{D}^\dagger = \mathcal{D}$) *

$\langle f|\mathcal{D}g\rangle = \langle \mathcal{D}f|g\rangle$

WHY WE LIKE HERMITIAN OPERATORS

$\langle y_i|y_j\rangle = \delta_{ij}$ *

$\lambda_i \in \mathbb{R}$

FOR DEGENERACIES ($\lambda_i = \lambda_j$), WE CAN USE GRAM-SCHMIDT ORTHOGONALIZATION TO ENSURE THAT * IS SATISFIED

TODAY:

GREEN'S FUNCTIONS FOR HERMITIAN OPERATORS

COMING UP:

STURM-LIOUVILLE EQUATIONS

** RHB BREAK THE DEFINITION OF THE INNER PRODUCT BY WRITING THE EIGENVALUE RELATION AS $\mathcal{L}|y_i\rangle = p(x)\lambda_i|y_i\rangle$.

* RHB DISTINGUISH BETWEEN SELF-ADJOINT AND HERMITIAN. IF THE ADJOINT IS DEFINED AS ABOVE, THERE IS NO DISTINCTION.

GREEN'S FUNCTIONS

SUPPOSE WE WANT TO SOLVE

$$\mathcal{L} u(x) = f(x)$$

FIRST, SOLVE THE EIGENVALUE PROBLEM FOR \mathcal{L} :

$$\mathcal{L} |y_i\rangle = \lambda_i |y_i\rangle$$

SINCE THE y_i ARE AN ORTHONORMAL BASIS, WRITE THE SOLUTION AS:

$$u(x) = \sum_i c_i y_i(x) \quad \star\star$$

TO DO: FIND THE c_i 'S (HOW?!))

HIT EQUATION $\star\star$ WITH \mathcal{L} , AND WE GET OUR ORIGINAL INHOMOGENEOUS ODE IN TERMS OF THE y_i :

$$\mathcal{L} u(x) = \mathcal{L} \sum_i c_i y_i(x)$$

$$f(x) = \sum_i c_i \lambda_i y_i(x)$$

NOW MULTIPLY BY $y_j^* \rho(x)$ AND INTEGRATE:

$$\int_a^b y_j^*(x) f(x) \rho(x) dx = \sum_i c_i \lambda_i \underbrace{\int_a^b y_j^*(x) y_i(x) \rho(x) dx}_{\delta_{ij}}$$

$$c_j = \frac{1}{\lambda_j} \int_a^b y_j^*(x) f(x) \rho(x) dx$$

DISCUSS

RETURNING TO EQUATION **

$$u(x) = \sum_i y_i(x) \frac{1}{\lambda_i} \int_a^b y_i^*(x') f(x') \rho(x') dx'$$

OR

$$|u\rangle = \sum_i \frac{1}{\lambda_i} |y_i\rangle \langle y_i | f \rangle$$

REARRANGE:

$$u(x) = \int_a^b \underbrace{\left[\sum_i \frac{1}{\lambda_i} y_i(x) y_i^*(x') \rho(x') \right]}_{G(x, x')} f(x') dx'$$

$G(x, x')$
GREEN'S FUNCTION!

QUESTION: IS $G(x, x')$ REAL?

A: PHYSICALLY, IT MUST BE REAL SINCE IT IS THE RESPONSE OF A PHYSICAL SYSTEM TO AN IMPULSE.

MATHEMATICALLY, WE CAN SEE FROM THE ABOVE EXPRESSION THAT $G(x, x') = G^*(x', x)$

EXERCISE: PROVE THAT $G(x, x')$ IS A SOLN. TO

$$\mathcal{L} G = \delta(x-x')$$

GIVEN: $G(x, x') = \sum_i \frac{1}{\lambda_i} y_i(x) y_i^*(x') \rho(x')$

SHOW: $DG = \delta(x-x')$

APPROACH: SHOW THAT $\int_a^b g(x') DG(x, x') dx' = g(x)$.

$$\begin{aligned} \int_a^b g(x') DG dx' &= \int_a^b g(x') D \left[\sum_i \frac{1}{\lambda_i} y_i(x) y_i^*(x') \rho(x') \right] dx' \\ &= \sum_i y_i(x) \int_a^b g(x') y_i^*(x') \rho(x') dx' \quad (\text{since } Dy_i = \lambda_i y_i) \\ &= \sum_i \langle x | y_i \rangle \langle y_i | g \rangle \\ &= \langle x | \sum_i | y_i \rangle \langle y_i | g \rangle \\ &= \langle x | g \rangle = g(x) \quad \underline{\underline{QED}} \end{aligned}$$