

SUMMARY

$$\mathcal{L}u(x) = f(x), \text{ WITH B.C.'S}$$

① IS  $\mathcal{L}$  HERMITIAN WITH THE GIVEN B.C.'S?

$$\langle \mathcal{L}\psi | \phi \rangle = \langle \psi | \mathcal{L}\phi \rangle$$

② SOLVE EIGENVALUE PROBLEM,

$$\mathcal{L}|y_i\rangle = \lambda_i |y_i\rangle$$

→ APPLY B.C.'S!

→ NORMALIZE!  $\langle y_i | y_i \rangle = 1$

→ RESOLVE ANY DEGENERACIES!  $\langle y_i | y_j \rangle = \delta_{ij}$

$$③ |u\rangle = \sum_i \frac{1}{\lambda_i} |y_i\rangle \langle y_i | f \rangle$$

$$u(x) = \int_a^b \left[ \sum_i \frac{1}{\lambda_i} y_i(x) y_i^*(x') \right] f(x') \rho(x') dx'$$

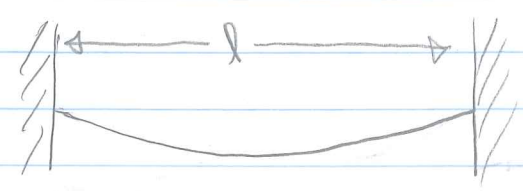
LAST TIME, WE USED ③ TO FIND THE SOLUTION TO

$$\mathcal{L}_x G(x, x'') = \delta(x - x'')$$

AND FOUND

$$G(x, x'') = \sum_i \frac{1}{\lambda_i} y_i(x) y_i^*(x'') \rho(x'').$$

EXAMPLE LOADED STRING



B.C.'s:  $y(0) = y(l) = 0$

FOR SMALL DEFLECTIONS  $y$ ,

$$\frac{d^2 y}{dx^2} = W(x)$$

ASIDE: WHAT IS THE EASY SOLN?

$$y(x) = \int dx \int dx W(x)$$

IS THE OPERATOR  $\frac{d^2}{dx^2}$ , WITH  $y(0) = y(l) = 0$ , HERMITIAN?

$$\langle \mathcal{D} f | g \rangle = \langle f | \mathcal{D} g \rangle$$

$$\int_0^l \frac{d^2 f^*}{dx^2} g dx = \int_0^l f^* \frac{d^2 g}{dx^2} dx$$

INTEGRATE BY PARTS...

~~$$g \frac{df^*}{dx} \Big|_0^l - \int_0^l \frac{df^*}{dx} \frac{dg}{dx} dx$$~~

~~$$= f^* \frac{dg}{dx} \Big|_0^l - \int_0^l \frac{df^*}{dx} \frac{dg}{dx} dx$$~~

B.C.'s :  $g(0) = g(l) = f^*(0) = f^*(l) = 0$

YES IT IS HERMITIAN.

★ NOTE HOW B.C.'S DETERMINE WHETHER  $\mathcal{D}$  IS ★  
HERMITIAN. SO  $\mathcal{D}$  MEANS THE  $\frac{d^2}{dx^2}$  + B.C.'S!

## EIGENVALUE PROBLEM:

$$\frac{d^2 y}{dx^2} = \lambda y$$

TRY  $y = e^{\alpha x}$

$$\alpha^2 = \lambda$$

~~$\alpha = \pm \sqrt{\lambda}$ , so~~ instead, let's put it this way:

$$y = A e^{\alpha x} + B e^{-\alpha x}$$

BC's:  $y(0) = 0 \Rightarrow A + B = 0$

$$y(l) = 0 \Rightarrow A e^{\alpha l} + B e^{-\alpha l} = 0$$

COMBINING EQ'S,  $A e^{\alpha l} - A e^{-\alpha l} = 0$

$$A (e^{\alpha l} - e^{-\alpha l}) = 0$$

IF  $A = 0$  THEN  $B = 0$  AND  $y = 0$  (UNACCEPTABLE!).

WE ARE LEFT WITH

$$e^{\alpha l} = e^{-\alpha l}$$

$$e^{2\alpha l} = 1 \xrightarrow{\text{allow } \alpha \in \mathbb{C}} 2\alpha l = i 2\pi n$$

$$\alpha = i\pi n / l.$$

$$2\alpha l = 0$$

$$\alpha = 0 \text{ ?!}$$

$$\Rightarrow \lambda = -\pi^2 n^2 / l^2$$

$n = 1, 2, 3, \dots$  (note that  $n=0$  does not satisfy the boundary conditions!)

SO OUR EIGENFUNCTION IS

$$Y = A e^{i\pi n x/l} - A e^{-i\pi n x/l}$$

$$= \underbrace{2Ai}_{\text{NORMALIZATION CONSTANT}} \sin(\pi n x/l)$$

$$Y = A \sin(\pi n x/l)$$

NORMALIZATION:

$$\langle Y | Y \rangle = \int_0^l A^2 \sin^2(\pi n x/l) dx = 1$$

$$A^2 \frac{l}{2} = 1$$

$$A = \sqrt{\frac{2}{l}}$$

$$Y_n = \sqrt{\frac{2}{l}} \sin(\pi n x/l)$$
$$\lambda_n = -\pi^2 n^2 / l^2$$

ANY DEGENERACIES?

→ NO.  $n = 1, 2, 3, \dots$



NOW, THE SOLUTION FOR ARBITRARY  $w(x)$  IS:

$$y(x) = \int_0^l \left[ \sum_n \frac{-l^2}{\pi^2 n^2} \frac{2}{l} \sin(\pi n x/l) \sin(\pi n x'/l) \right] w(x') dx'$$

$$= \int_0^l \left[ \sum_n \frac{-l}{\pi^2 n^2} \sin(2\pi n x/l) \sin(2\pi n x'/l) \right] w(x') dx'$$

CHECK:

$$y''(x) = \int_0^l \left[ \sum_n \frac{2}{l} \sin(\pi n x/l) \sin(\pi n x'/l) \right] w(x') dx'$$

$$y''(x) = \langle x | \underbrace{\left( \sum_n |y_n\rangle \langle y_n| \right)}_{\text{IDENTITY}} | w \rangle \rangle \checkmark = \langle x | w \rangle = w(x)$$

AS LONG AS THE  $y_n$  ARE COMPLETE, IT WORKS

THE  $y_n$  WILL BE COMPLETE IF  $\mathcal{L}$  IS HERMITIAN.

(THIS IS A PROPERTY OF HERMITIAN OPERATORS THAT WE WILL NOT PROVE)

CHECK THE GREEN'S FUNCTION:

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See examples/eigenfunction-expansion/

$$y = \sum_{k=1}^{50} \left( -\frac{2}{\pi^2 k^2} \sin(\pi k x) \right) \sin(\pi k n)$$

$(\lambda = 1)$

$x < 0$

$x > 1$

