

§ 17.4 STURM-LIOUVILLE EQUATIONS

MANY ODEs THAT COMMONLY OCCUR IN PHYSICS CAN BE PUT IN STURM-LIOUVILLE FORM:

$$p(x) \frac{d^2 y}{dx^2} + \frac{dp}{dx} \frac{dy}{dx} + q(x)y + \lambda p(x)y = 0$$

where $p(x)$ and $q(x)$ are real functions.

NOTE THE INVERTED SIGN OF λ — THIS IS RHAB'S CONVENTION

EIGENVALUE

WEIGHTING FACTOR FOR NORMALIZATION OF EIGENFUNCTIONS

THE STURM-LIOUVILLE OPERATOR IS THEN \mathcal{L} WHERE

$$\mathcal{L} y = \lambda y$$

$$\mathcal{L} = \frac{p}{p} \frac{d^2}{dx^2} - \frac{p'}{p} \frac{d}{dx} - \frac{q}{p}$$

$$= - \left[\frac{1}{p} \frac{d}{dx} \left(p \frac{d}{dx} \right) + \frac{q}{p} \right]$$

IS \mathcal{L} HERMITIAN?

SHOW: $\langle \mathcal{L}f | g \rangle = \langle f | \mathcal{L}g \rangle$

I.E., SHOW $\langle \mathcal{L}f | g \rangle$ INVARIANT UNDER $f^* \leftrightarrow g$.

$$\int_a^b -\frac{1}{p} \left[\frac{d}{dx} \left(p \frac{df^*}{dx} \right) + q f^* \right] g \underbrace{p dx}_{dV}$$

$$= -\int_a^b \frac{d}{dx} \left(p \frac{df^*}{dx} \right) g dx - \int_a^b q f^* g dx$$

↑
SYMMETRIC
UNDER $f^* \leftrightarrow g$

FIRST TERM:

$$\int u dv = uv - \int v du$$

$$u = g, \quad du = g' dx; \quad dv = \frac{d}{dx} \left(\right) dx, \quad v = \left(\right)$$

$$g \left(p \frac{df^*}{dx} \right) \Big|_a^b + \int_a^b p f^{*'} g' dx$$

↑

↑
SYMMETRIC UNDER
 $f^* \leftrightarrow g$

THE WHOLE EXPRESSION WILL BE SYMMETRIC IF THIS TERM IS ZERO.

$$g p f^{*'} \Big|_a^b = 0 \iff \mathcal{L} \text{ IS HERMITIAN}$$

EIGENFUNCTION METHODS FOR DIFFERENTIAL EQUATIONS

Equation	$p(x)$	$q(x)$	λ	$\rho(x)$
Hypergeometric	$x^c(1-x)^{a+b-c+1}$	0	$-ab$	$x^{c-1}(1-x)^{a+b-c}$
Legendre	$1-x^2$	0	$\ell(\ell+1)$	1
Associated Legendre	$1-x^2$	$-m^2/(1-x^2)$	$\ell(\ell+1)$	1
Chebyshev	$(1-x^2)^{1/2}$	0	v^2	$(1-x^2)^{-1/2}$
Confluent hypergeometric	$x^c e^{-x}$	0	$-a$	$x^{c-1} e^{-x}$
Bessel*	x	$-v^2/x$	α^2	x
Laguerre	$x e^{-x}$	0	v	e^{-x}
Associated Laguerre	$x^{m+1} e^{-x}$	0	v	$x^m e^{-x}$
Hermite	e^{-x^2}	0	$2v$	e^{-x^2}
Simple harmonic	1	0	ω^2	1

Table 17.1 The Sturm-Liouville form (17.34) for important ODEs in the physical sciences and engineering. The asterisk denotes that, for Bessel's equation, a change of variable $x \rightarrow x/a$ is required to give the conventional normalisation used here, but is not needed for the transformation into Sturm-Liouville form.

GIVEN A MORE GENERAL EQUATION,

$$a(x)y'' + b(x)y' + c(x)y + \lambda d(x)y = 0$$

CAN IT BE EXPRESSED IN STURM-LIOUVILLE FORM?

RHB CLAIMS WE NEED ONLY MULTIPLY BY ...

$$F(x) \equiv \exp\left(\int \frac{b(x) - a'(x)}{a(x)} dx\right) \quad (\text{cf. Eq. 17.39})$$

HOW CAN WE TEST THIS?

$$p(x) = a(x) F(x) \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\}$$

$$p'(x) \stackrel{?}{=} b(x) F(x)$$

$$q(x) = c(x) F(x)$$

$$r(x) = d(x) F(x)$$

CHECK

$$p' = a'F + aF'$$

$$F' = \frac{b-a'}{a} F, \text{ so}$$

$$p' = a'F + (b-a')F = bF \checkmark$$

BUT NOTE: THERE IS OFTEN A SIMPLER OR MORE OBVIOUS WAY TO GET $F(x)$.

EXAMPLE ELECTROSTATICS IN CYLINDRICAL SYMMETRY

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$\nabla^2 \Phi = \frac{\rho}{\epsilon_0}$ BECOMES

$$\left(\frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \right) \right) \Phi = \frac{\rho}{\epsilon_0} \equiv f(r)$$

$$\Phi'' + \frac{1}{r} \Phi' = f(r)$$

EIGENVALUE PROBLEM:

$$\Phi'' + \frac{1}{r} \Phi' + \lambda \Phi = 0$$

Q: PUT INTO S-L FORM --- HOW?

A: MULTIPLY BY r .

$$r \Phi'' + \Phi' + r \lambda \Phi = 0$$

\uparrow \uparrow
 $p(r) = r$ $p(r) = r$

$$q(r) = 0!$$

Q: DOES THIS RESEMBLE ANYTHING IN TABLE 17.1? ...

* NOTE HOW THE $p(r) = r$ IS NATURALLY INCLUDED IN ∇^2 . THIS IS NORMAL!

A: TURNS OUT WE HAVE BESSEL'S EQUATION WITH $\nu = 0$;

→ BUT HOW DOES THE WHOLE EIGENVALUE THING WORK?

BESSEL'S EQN IS: (P. 568)

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0, \text{ SOLN. } y = J_\nu(x)$$

LET $\bar{x} = x/\alpha$

$$\bar{x} y''(\alpha \bar{x}) + y'(\alpha \bar{x}) - \frac{\nu^2}{\bar{x}} y(\alpha \bar{x}) + \alpha^2 \bar{x} y(\alpha \bar{x}) = 0$$

(PRIME = $\frac{d}{d\bar{x}}$!) SOLN. $y = J_\nu(\alpha \bar{x})$

NOW $\bar{x} \rightarrow r$

$$y \rightarrow J_\nu(\alpha r) = \Phi(r)$$

AND WE HAVE $\nu = 0$,

$$\lambda = \alpha^2$$

NOTE THAT THERE ARE A CONTINUUM OF EIGENVALUES α^2 . SO

$$\sum_i \rightarrow \int$$

ORTHONORMALITY: § 18.5.3

CAREFUL WITH NORMALIZATION — SHOULD HAVE A $1/\alpha$ IN IT!