

# § 20.1 — SOME IMPORTANT PDE'S

## WAVE EQUATION

$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

VERIFY THAT THE FOLLOWING IS A SOLN:

$$u(\vec{r}, t) = f(\hat{k} \cdot \vec{r} - ct) \quad (\text{IN-CLASS EXERCISE NEXT TIME})$$

SIMPLER CASE:

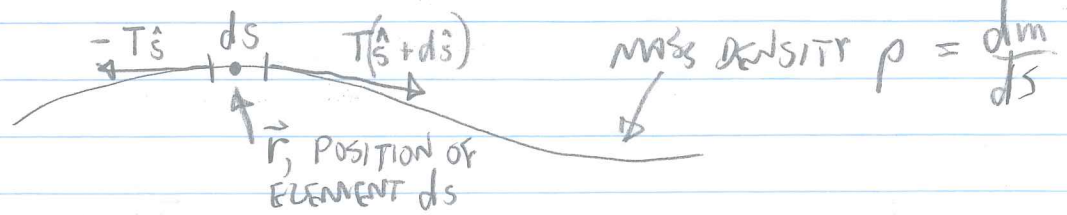
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

TRY  $u(x, t) = f(x \pm ct)$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left( f'(x \pm ct) \right) \\ &= f''(x \pm ct) \end{aligned}$$

$$\begin{aligned} \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} &= \frac{1}{c^2} \frac{\partial}{\partial t} \left( \pm c f'(x \pm ct) \right) \\ &= \frac{1}{c^2} c^2 f''(x \pm ct) = f''(x \pm ct) \end{aligned}$$

INTERPRETATION: A DISTURBANCE OF ANY SHAPE  $f(x)$  CAN PROPAGATE TO THE LEFT OR TO THE RIGHT AT SPEED  $c$ .

EXAMPLESTRING,  
TENSION  $T$ .

$$\text{EOM: } dm \frac{d^2 \vec{r}}{dt^2} = -T \hat{s} + T(\hat{s} + d\hat{s})$$

$$\text{BUT } dm = \rho ds$$

$$\rho \frac{d^2 \vec{r}}{dt^2} = T \frac{d\hat{s}}{ds} \quad \star$$

NOW DESCRIBE STRING BY  $y(x)$  SO THAT

$$ds = \sqrt{dx^2 + dy^2} = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

TO FIRST ORDER IN  $y$  (AND  $y'$ ,  $y''$ , ...),

$$ds = dx.$$

$$\text{SIMILARLY, } \hat{s} = \frac{dx \hat{x} + dy \hat{y}}{\sqrt{dx^2 + dy^2}} = \frac{(\hat{x} + y' \hat{y}) dx}{dx}$$

$$= \hat{x} + y' \hat{y}$$

AND SO

$$\frac{d\hat{s}}{ds} = \frac{d\hat{s}}{dx} = y'' \hat{y}$$

SO  $\star$  BECOMES

$$\rho \frac{d^2 \vec{r}}{dt^2} = \rho \frac{d^2 (x \hat{x} + y \hat{y})}{dt^2} = T \frac{\partial^2 y}{\partial x^2} \hat{y}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 y}{\partial x^2}$$

NOTE  $c^2 = T/\rho$ 

$$c = \sqrt{T/\rho}$$

# DIFFUSION EQUATION

$$k \nabla^2 u = \frac{\partial u}{\partial t}$$

↑  
LIKE WAVE EQN., BUT FIRST-ORDER IN TIME  $t$ .

Q: HOW DO SOLN'S TO THE DIFFUSION & WAVE EQNS. DIFFER?

A: DIFFUSION EQN. HAS NO INERTIA. ANALOGY:  $m\ddot{x} = -kx - \mu\dot{x}$

# LAPLACE'S EQUATION & POISSON'S EQN.

$$\nabla^2 u = 0 \quad \text{LAPLACE}$$

$$\nabla^2 u = \rho(\vec{r}) \quad \text{POISSON}$$

# ADVECTION EQN.

$$\frac{\partial u}{\partial t} = -\vec{v}(\vec{r}, t) \cdot \nabla u - u \nabla \cdot \vec{v}$$

HOW DERIVED? THINK OF  $\int u dV$  AS A CONSERVED QUANTITY. FLUX OF  $u$ -NESS:

$$\vec{F}_u \equiv u \vec{v} \quad \dots \quad u\text{-NESS CARRIED PASSIVELY BY } \vec{v}(\vec{r}, t)$$

CONSERVED IN THE SENSE THAT

$$\textcircled{3} \nabla \cdot \vec{F}_u = -\frac{\partial u}{\partial t}$$

← COMPARE →

$$\textcircled{2} \int_V \nabla \cdot \vec{F}_u dV = - \int_V \frac{\partial u}{\partial t} dV$$
  
$$\textcircled{1} \oint_S \vec{F}_u \cdot d\vec{s} = - \int_V \frac{\partial u}{\partial t} dV$$

$$\textcircled{4} \frac{\partial u}{\partial t} = -(\vec{v} \cdot \nabla u + u \nabla \cdot \vec{v})$$

SCHRÖDINGER'S EQUATION

$$-\frac{\hbar^2}{2m} \nabla^2 u + V(\vec{r}) u = i\hbar \frac{\partial u}{\partial t}$$

$u \in \mathbb{C}$ , UNLIKE OUR OTHER PDE'S!