

# CH. 13 - FOURIER TRANSFORMS

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \equiv \tilde{f}(k)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \quad (\text{UNIQUE INVERSE})$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-ik'x} dx = \delta(k-k') \quad (\text{ORTHOGONALITY})$$

$$\mathcal{F}[f'(t)] = i\omega \tilde{f}(\omega) \leftarrow \begin{array}{|l} \text{ALLOWS SOLUTION OF D.E.'S} \\ \text{WITH CONSTANT COEFF'S} \end{array}$$

$$\mathcal{F}\left[\int f(t) dt\right] = \frac{1}{i\omega} \tilde{f}(\omega) + 2\pi c \delta(\omega)$$

↑  
FOURIER TRANSFORM OF THE  
CONSTANT OF INTEGRATION.

$$\mathcal{F}[f(at)] = \frac{1}{a} \tilde{f}\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}[f(t+a)] = e^{i\omega a} \tilde{f}(\omega)$$

$$\mathcal{F}[e^{\alpha t} f(t)] = \tilde{f}(\omega + i\alpha) \quad (\alpha \in \mathbb{C})$$

EVEN(f)  $\xrightarrow{\mathcal{F}}$  EVEN( $\omega$ )

ODD(f)  $\xrightarrow{\mathcal{F}}$   $i$  ODD( $\omega$ )

$$f * g \equiv \int_{-\infty}^{\infty} f(x) g(z-x) dx = g * f$$

↓  
MINUS!

$$\mathcal{F}(f+g) = \sqrt{2\pi} \tilde{f} \tilde{g} \quad ; \quad \mathcal{F}(fg) = \frac{1}{\sqrt{2\pi}} \tilde{f} * \tilde{g}$$

CORRELATION:

$$f \otimes g = \int_{-\infty}^{\infty} f^*(x) g(x+z) dx = [g \otimes f]^*(-z)$$

$$\mathcal{F}(f \otimes g) = \sqrt{2\pi} \tilde{f}^* \tilde{g}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk$$

# §15.2 GREEN'S FUNCTIONS FOR ODE'S

ODE:  $\mathcal{L}y(t) = f(t)$  on  $a < t < b$

WE REPRESENT  $y(t) = \int_a^b G(t, t') f(t') dt'$   $\star$  SOLN.

(AND  $f(t) = \int_a^b \delta(t-t') f(t') dt'$ )

WHICH LEADS NATURALLY TO

$\mathcal{L}G(t, t') = \delta(t-t')$   $\star\star$

SOLUTION FOR G:

- ① FIND GEN SOLN OF  $\mathcal{L}y_c = 0$
- ②  $G = \begin{cases} G_1 & t < t' \\ G_2 & t > t' \end{cases}$  CONSTRUCTED FROM  $y_c$ .
- ③ INTEGRATE  $\star\star$ :

$\int_{t'-\epsilon}^{t'+\epsilon} dt [\mathcal{L}G(t, t')] = 1$

→ ASSUME ALL BUT HIGHEST DERIVATIVE TERM GOES TO ZERO IN LIMIT  $\epsilon \rightarrow 0$ .

THESE CONSTRAINTS  $\left\{ \begin{array}{l} G_1 \text{ AND } G_2 \\ \text{MATCHING} \\ \text{ACROSS } t=t' \end{array} \right\}$

HIGHEST ORDER TERM IN  $\mathcal{L}G$  HAS  $\delta(t-t')$   
 (that is, the next highest derivative)  
 2<sup>nd</sup> HIGHEST TERM IN  $\mathcal{L}G$  HAS  $H(t-t')$   
 LOWER TERMS ARE CONTINUOUS!

④ SOLUTION IS  $\star$ .

# CH. 17 — ODE EIGENFUNCTION METHODS

SOLVE ODE'S OF FORM

$$\mathcal{D}y(x) = f(x). \quad \left( \mathcal{D} |y\rangle = |f\rangle \right)$$

BEGIN BY SOLVING EIGENVALUE PROBLEM,

$$\mathcal{D}y_n(x) = \lambda_n y_n(x)$$

$$\text{IF } \langle y_m | y_n \rangle \equiv \int_a^b y_m^*(x) y_n(x) \rho(x) dx = \delta_{mn},$$

(OR  $\langle y_k | y_{k'} \rangle = \delta(k-k')$ ), AND IF THE SET OF  $|y_n\rangle$  IS COMPLETE, THEN

$$|y\rangle = \sum_n \frac{1}{\lambda_n} |y_n\rangle \langle y_n | f \rangle$$

$$\text{I.E. } y(x) = \int_a^b \left( \sum_n \frac{1}{\lambda_n} y_n^*(x') y_n(x) \right) f(x') \rho(x') dx'$$

HERMITIAN OPERATORS → GUARANTEED ORTHONORMAL BASIS OF EIGENFUNCTIONS! (AND  $\lambda_n \in \mathbb{R}$ )

TO FIND OUT IF HERMITIAN:

$$\langle f | \mathcal{D} | g \rangle = \left[ \langle g | \mathcal{D} | f \rangle \right]^* \quad \text{WHERE } f \& g \text{ OBEY } \boxed{\text{B.C.'s}}$$

$$\text{I.E. } \int_a^b f^*(x) [\mathcal{D} g(x)] \rho(x) dx = \int_a^b [\mathcal{D} f(x)]^* g(x) \rho(x) dx$$

STURM-LIOUVILLE OPERATORS

$$\mathcal{D} = \frac{-1}{\rho(x)} \left[ p(x) \frac{d^2}{dx^2} + p'(x) \frac{d}{dx} + q(x) \right]$$

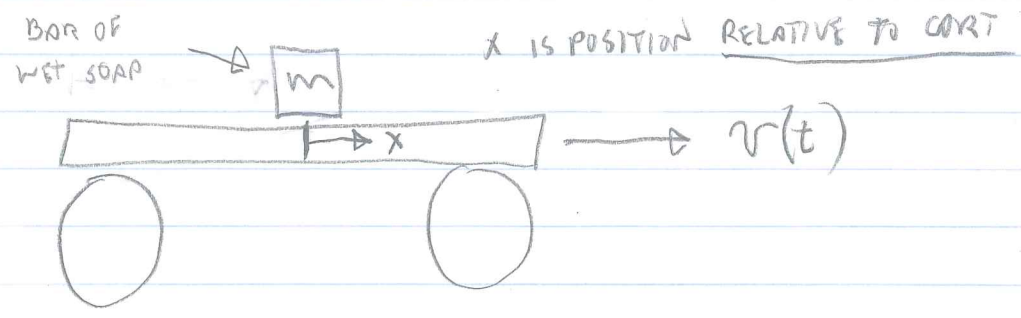
(NOTE SIGN CONVENTION — EIGENVALUES MINUS WHAT YOU MIGHT THINK)

$$\mathcal{D}y(x) = \lambda y(x)$$

$\mathcal{D}$  IS HERMITIAN IFF B.C.'S LEAD TO

$$\left[ y_n^* p y_m' \right]_{x=a}^b = 0$$

### EXAMPLE — G.F., FOURIER TRANSFORMS



$$ma = -\mu \dot{x}$$

INITIAL CONDITIONS:  $\dot{x} = 0, x = 0$

$a = \ddot{x} + \dot{v}$  IN LAB (INERTIAL) FRAME

$$m\ddot{x} + \mu\dot{x} = -\dot{v}(t)$$

① FIND GREEN'S FUNCTION

② SUPPOSE  $v(t) = \begin{cases} v_0, & 0 < t < t_0 \\ 0, & \text{ELSEWHERE} \end{cases}$

FIND  $x(t)$ .

③ SOLVE EQUATION BY FOURIER TRANSFORM, TO FIND  $\tilde{x}$  FOR A GIVEN  $\tilde{v}$ .

④ FIND  $\tilde{v}$ , <sup>FROM ②</sup> AND USE THE FOURIER SOLN TO FIND  $\tilde{x}$ .

⑤ TAKE THE INVERSE TRANSFORM TO CALCULATE  $x(t)$  FROM  $\tilde{x}$ .

(THIS IS TOO TRICKY FOR AN EXAM)

① G.F.:

FIRST, SOLVE COMPLEMENTARY EQN:

$$\ddot{x}_c + \frac{\mu}{m} \dot{x}_c = 0$$

$$\text{LET } x_c = A e^{\alpha t}$$

$$\alpha^2 + \frac{\mu \alpha}{m} = 0$$

$$\alpha = \frac{-\frac{\mu}{m} \pm \sqrt{\left(\frac{\mu}{m}\right)^2 - 0}}{2}$$

$$\alpha = 0, \frac{\mu}{m}$$

WE WANT G TO SATISFY

$$\ddot{G} + \frac{\mu}{m} \dot{G} = \delta(t-t')$$

$$\text{LET } G = \begin{cases} G_1 = A + B e^{-\mu t/m}, & t < t' \\ G_2 = C + D e^{-\mu t/m}, & t > t' \end{cases}$$

$$\bullet \text{ INIT COND'S } x=0, \dot{x}=0 \Rightarrow A=B=0$$

$$\bullet \int_{t \neq t'} \left( \ddot{G} + \frac{\mu}{m} \dot{G} \right) dt = 1$$

$$\Rightarrow \dot{G} \Big|_{t=t'-\epsilon}^{t'+\epsilon} = 1 \Rightarrow -\frac{\mu}{m} D e^{-\mu t'/m} = 1$$

$$\Rightarrow D = -\frac{m}{\mu} e^{\mu t'/m}$$

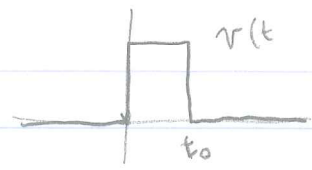
$$\bullet \text{ CONTINUITY OF } G \text{ AT } t=t':$$

$$C + D e^{-\mu t'/m} = 0 \Rightarrow C = \frac{m}{\mu}$$

$$\therefore G = \begin{cases} 0, & t < t' \\ \frac{m}{\mu} (1 - e^{-\mu(t-t')/m}), & t > t' \end{cases}$$

Q: WHAT IF WE HAD  $\int^{\text{ST}}$  ORDER ODE? DISCUSS.

②  $v(t) = \begin{cases} v_0, & 0 < t < t_0 \\ 0, & \text{ELSE} \end{cases}$



$$x = \int_{-\infty}^{\infty} G(t, t') \dot{v}(t') dt'$$

$$\dot{v}(t) = v_0 \delta(t) - v_0 \delta(t - t_0) \quad (\text{DISCUSS!})$$

$$x = \begin{cases} 0, & t < 0 \\ \frac{v_0 m}{m} (1 - e^{-\mu t/m}), & 0 < t < t_0 \\ \frac{v_0 m}{m} (1 - e^{-\mu t/m}) - \frac{v_0 m}{m} (1 - e^{-\mu(t-t_0)/m}), & t > t_0 \end{cases}$$

③  $\ddot{x} + \frac{m\dot{x}}{m} = -\dot{v}$

$$-\omega^2 \tilde{x} + i\omega \frac{m}{m} \tilde{x} = -i\omega \tilde{v}$$

$$\tilde{x} = \frac{-i\omega \tilde{v}}{-\omega^2 + i\omega \frac{m}{m}} = \frac{-\tilde{v}}{\frac{m}{m} + i\omega}$$

④  $\tilde{v} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} [v_0 \delta(t) - v_0 \delta(t - t_0)] dt$

$$= \frac{v_0}{\sqrt{2\pi}} [1 - e^{-i\omega t_0}]$$

$$\tilde{x} = \frac{v_0}{2\pi} \frac{[e^{-i\omega t_0} - 1]}{\frac{m}{m} + i\omega} = \frac{v_0}{2\pi} \frac{[e^{-i\omega t_0} - 1]}{\frac{m}{m} + i\omega}$$

ETC

# EIGENFUNCTION METHODS EXAMPLE

①

CONSIDER THE DIFFERENTIAL EQUATION ON  $0 \leq x \leq L$ ,

$$a^2 \frac{d^2 y}{dx^2} + y = R(x), \quad \frac{dy}{dx} = 0 \text{ AT } x=0, L.$$

(a) SHOW THAT  $\mathcal{L}_\lambda$  IS HERMITIAN (HINT: PUT IT INTO STURM-LIOUVILLE FORM)

→ IT'S ALREADY IN STURM LIOUVILLE FORM:

$$a^2 \frac{d^2 y}{dx^2} + 0 \frac{dy}{dx} + y = R(x)$$

→ THE B.C.'S MEET THE HERMITICITY CONDITION FOR S-L OPERATORS.

(b) FIND THE EIGENVALUES & EIGENFUNCTIONS OF THE OPERATOR WITH THE GIVEN B.C.'S (SINES AND COSINES WOULD BE A GOOD PLACE TO START)

$$a^2 \frac{d^2 y_i}{dx^2} + y_i = \lambda_i y_i$$

$$\text{TRY } y_i = A \sin(k_i x) + B \cos(k_i x)$$

PLUG IN:

$$-a^2 k_i^2 y_i + y_i = \lambda_i y_i$$

$$\lambda_i = 1 - a^2 k_i^2$$

BC'S:

$$A=0, \quad k_n L = n\pi, \quad n=0, 1, 2, \dots$$

$$\text{NORMALIZATION: } \int_0^L B^2 \cos^2(n\pi x/L) dx = 1$$

$$n > 0: \quad B^2 \frac{L}{2} = 1 \Rightarrow B_n = \sqrt{\frac{2}{L}} \rightarrow$$

$$n = 0: \quad B^2 L = 1 \Rightarrow B_0 = \sqrt{\frac{1}{L}}$$

(2)

SO WE HAVE

$$\lambda_n = 1 - a^2 k_n^2 = 1 - a^2 n^2 \pi^2 / L^2$$

$$Y_n = \begin{cases} \sqrt{\frac{1}{L}}, & n=0 \\ \sqrt{\frac{2}{L}} \cos(n\pi x/L), & n=1, 2, 3, \dots \end{cases}$$

(c) EXPRESS THE SOLUTION TO THE INHOMOGENEOUS PROBLEM IN TERMS OF THE EIGENFUNCTIONS  $Y_n(x)$

$$\mathcal{D}^2 |y\rangle = |R\rangle$$

$$\mathcal{D}^2 |Y_n\rangle = \lambda_n |Y_n\rangle$$

$$|R\rangle = \sum_n C_n |Y_n\rangle$$

$$= \sum_n |Y_n\rangle \langle Y_n | R \rangle$$

$$\mathcal{D}^2 \left[ \sum_n \frac{1}{\lambda_n} |Y_n\rangle \langle Y_n | R \rangle \right] = \sum_n |Y_n\rangle \langle Y_n | R \rangle = |R\rangle$$

$$\therefore |y\rangle = \sum_n \frac{1}{\lambda_n} |Y_n\rangle \langle Y_n | R \rangle$$

$$y(x) = \sum_n \left[ \frac{Y_n(x)}{1 - a^2 n^2 \pi^2 / L^2} \int_0^L Y_n(x') R(x') dx' \right]$$

$$\text{OR } y(x) = \int_0^L \left( \sum_n \frac{Y_n(x) Y_n(x')}{1 - a^2 n^2 \pi^2 / L^2} \right) R(x') dx'$$

(d) WHAT MIGHT GO WRONG IF  $\frac{L}{\pi a}$  HAPPENS TO BE AN INTEGER? IS THERE A WAY TO HANDLE THIS?!