

WE ARE ON CH. 20.

LAST TIME:

DISCUSSED VARIOUS PHYSICS PDES AND WHERE
THEY COME FROM.

THIS TIME:

• §§ 20.2-20.3.1, FIRST-ORDER EQUATIONS
(A PRELUDE TO THE METHOD OF CHARACTERISTICS)

NEXT TIME:

§ 20.3.2 INHOMOGENEOUS FIRST-ORDER

§ 20.3.3 SECOND-ORDER EQUATIONS

§ 20.2 (MY PRESENTATION IS SIMILAR TO RHB, BUT THEY GIVE A DIFFERENT EXAMPLE) ①

CONSIDER THE FIRST-ORDER ODE

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0. \quad \star$$

I ASSERT THAT THE FOLLOWING ARE PARTICULAR SOLUTIONS:

$$u_1 = \frac{x^2 + y^2}{xy}$$

$$u_2 = \ln y - \ln x$$

$$u_3 = \theta, \quad \text{WHERE } x = r \cos \theta, \quad y = r \sin \theta$$

$$u_4 = \frac{z}{|z|}, \quad \text{WHERE } z = x + iy.$$

→ DIVIDE INTO 3-4 GROUPS TO CHECK THESE SOLUTIONS.

QUESTION: WHAT DO THESE FOUR SOLUTIONS HAVE IN COMMON?

ANSWER: ALL ARE OF THE FORM

$$u = f(p), \quad \text{WHERE } p(x, y) = y/x.$$

THIS IS, IN FACT, THE GENERAL SOLUTION FOR $\star!$

HOW DID I CONTRIVE THIS EXAMPLE? RHB'S RECIPE:

- CHOOSE $p(x,y) = y/x$. ANY FUNCTION WILL DO!
(BOOK USES x^2+2y AS AN EXAMPLE.)
- LETTING $u = f(p)$ FOR SOME ARBITRARY f ,

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= f' \frac{\partial p}{\partial x} = \frac{-y}{x^2} f' \\ \frac{\partial u}{\partial y} &= f' \frac{\partial p}{\partial y} = \frac{f'}{x} \end{aligned} \right\} \star\star$$

- WE WANT AN ODE THAT WORKS FOR ANY $f(p)$.
TO ELIMINATE f' FROM THE ABOVE, DIVIDE
THE TWO EQUATIONS (OR CROSS-MULTIPLY THEM):

$$\frac{\partial u}{\partial x} \cdot \cancel{f'} \frac{\partial p}{\partial y} = \frac{\partial u}{\partial y} \cdot \cancel{f'} \frac{\partial p}{\partial x}$$

WHICH, IN OUR CASE, IS

$$\frac{1}{x} \frac{\partial u}{\partial x} + \frac{y}{x^2} \frac{\partial u}{\partial y} = 0 \xrightarrow{\cdot x^2} \boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0}$$

- IT ONLY REMAINS TO CHOOSE SOME PARTICULAR SOLUTIONS
FOR THE EXAMPLE:

$$u_1 = p + \frac{1}{p} = \frac{y}{x} + \frac{x}{y} = \frac{x^2+y^2}{xy}$$

$$u_2 = \ln\left(\frac{y}{x}\right) = \ln y - \ln x$$

$$u_3 = \text{TAN}^{-1}\left(\frac{y}{x}\right) = \theta \quad (\text{IN POLAR COORD'S})$$

$$u_4 = e^{i\theta} = \cos \theta + i \sin \theta = \frac{x+iy}{\sqrt{x^2+y^2}} = \frac{z}{|z|}$$

§20.3.1 SOLVING FIRST-ORDER EQ'S (2 INDEPENDENT VAR'S)

GENERAL FORM:

$$A(x,y) \frac{\partial u}{\partial x} + B(x,y) \frac{\partial u}{\partial y} + C(x,y) u = R(x,y) \quad (20.9)$$

NOTE: IF EITHER OF THESE IS ZERO, IT'S AN ODE. BUT THE "CONSTANT" OF INTEGRATION IS AN ARBITRARY FUNCTION OF x (IF $A=0$) OR OF y (IF $B=0$). (SEE EXAMPLE IN TEXT)

FIRST, CONSIDER $C=R=0$. IF $u=f(p)$, THEN (AS IN OUR EXAMPLE, EX. **),

$$\frac{\partial u}{\partial x} = f' \frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = f' \frac{\partial p}{\partial y}$$

SUB INTO (20.9):

$$A \cancel{f'} \frac{\partial p}{\partial x} + B \cancel{f'} \frac{\partial p}{\partial y} = 0$$

$$A \frac{\partial p}{\partial x} + B \frac{\partial p}{\partial y} = 0, \quad p = p(x,y) \quad (20.10)$$

GOT OUR ORIGINAL PDE BACK. WHAT GOOD IS THAT?!

SEEMINGLY INNOCENT QUESTION:

WHAT DOES IT TAKE TO HOLD p (AND THUS f) CONSTANT WHILE x AND y CHANGE?

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = 0 \quad (20.11)$$

COMBINING 20.10 WITH 20.11 (DIVIDE EQ'S),

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)} \quad (20.12)$$

IF WE INTEGRATE THIS EQ., THEN WE WILL GET AN INTEGRATION CONSTANT, C :

$$\text{BLAH}(x,y) = C$$

ANSWER:

IF $\text{BLAH}(x,y)$ IS CONSTANT, THEN p WILL BE CONSTANT. WE THEREFORE MAKE THE IDENTIFICATION

$$p = \text{BLAH}(x,y)$$

OR, IF MORE CONVENIENT,

$$p = \text{FUNCTION}(\text{BLAH}(x,y)).$$

DISCUSS: THE ARGUMENT ON THIS PAGE MAY SEEM MORE COMPELLING IF IT IS REVIEWED IN REVERSE ORDER.

NOTE: THIS PROCEDURE IS NOT GUARANTEED TO WORK. NOT ALL LINEAR PDE'S HAVE SOLN'S OF FORM $f(p)$.

EXAMPLE (BOOK HAS A DIFFERENT ONE)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \quad (\text{SAME PDE AS BEFORE})$$

FIRST, LET'S DO IT WITHOUT B.C.'S, TO GET A GENERAL SOLN.

APPLY EQ. 20.12:

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$C + \ln x = \ln y$$

$$C = \ln y - \ln x = \ln \left(\frac{y}{x} \right)$$

WE COULD LET $p = \ln \left(\frac{y}{x} \right)$, BUT

FOR SIMPLICITY, LET

$$p = \frac{y}{x}$$

$$\text{GEN SOLN: } u = f(p) = f\left(\frac{y}{x}\right)$$

QUESTIONS?

→ NOW APPLY B.C. : $u = 4y + 1$ ON THE LINE $x = 1$.

EXAMPLE (YOU TRY!)

FIND GENERAL SOLUTION OF

$$\frac{\partial u}{\partial x} = x \frac{\partial u}{\partial y}$$

(ANSWER: $u(x,y) = x^2 + 2y$)

BOUNDARY CONDITIONS:

(i) $u = y$ on $x = 1$

(ii) $u = y$ on $y = -x^2$ (SLIGHTLY TRICKIER!)