

§§ 20.3.1 - 20.3.2

①

LAST TIME WE CONSIDERED EQUATIONS OF THE FORM

$$A(x,y) \frac{\partial u}{\partial x} + B(x,y) \frac{\partial u}{\partial y} = 0.$$

WE ASSUMED A SOLN. OF THE FORM

$$u(x,y) = f(p), \text{ FOR SOME } p(x,y).$$

→ THIS PRODUCED A PDE FOR p , WITHOUT f IN IT. (KEY POINT)

WE FOUND $p(x,y)$ BY SETTING $dp=0$ AND FINDING

$$\frac{dx}{A} = \frac{dy}{B}$$

↓ INTEGRATE

$$\text{BLAH}(x,y) = \text{CONST.} = p$$

$$\text{OR } p = F(\text{BLAH}) \text{ FOR CONVENIENCE.}$$

TODAY WE CONSIDER MORE GENERAL PROBLEMS:

$$A(x,y) \frac{\partial u}{\partial x} + B(x,y) \frac{\partial u}{\partial y} + C(x,y) u = R(x,y)$$

FIRST ↗

SECOND ↗

ADD A TERM $C(x, y)u$:

$$A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + Cu = 0 \quad \star$$

TRY OUR OLD TRICK:

$$u(x, y) = f(p(x, y))$$

CROSS
OUT
WHEN
DONE

$$\frac{\partial u}{\partial x} = f' \frac{\partial p}{\partial x}, \quad \frac{\partial u}{\partial y} = f' \frac{\partial p}{\partial y}$$

SO OUR PDE (\star) BECOMES:

$$A f' \frac{\partial p}{\partial x} + B f' \frac{\partial p}{\partial y} + C f(p) = 0$$

UH-OH! WE CAN'T GET RID OF f . THIS IS A NO-GO.

NEW TRICK: $u(x, y) = h(x, y) f(p)$

$$\text{NOW, } \frac{\partial u}{\partial x} = \frac{\partial h}{\partial x} f(p) + h f' \frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial h}{\partial y} f(p) + h f' \frac{\partial p}{\partial y}$$

SUBSTITUTING INTO \star , OUR PDE BECOMES

$$\left(A \frac{\partial h}{\partial x} + B \frac{\partial h}{\partial y} + Ch \right) f(p) + \left(A \frac{\partial p}{\partial x} + B \frac{\partial p}{\partial y} \right) h f' = 0$$

IF $h(x, y)$ IS ANY NONZERO SOLN. OF \star , THIS TERM DISAPPEARS.

WE ARE LEFT WITH A C-LESS PDE:

$$A \frac{\partial p}{\partial x} + B \frac{\partial p}{\partial y} = 0$$

... WHICH WE KNOW HOW TO SOLVE:

$$\frac{dx}{A} = \frac{dy}{B}$$

↓ INTEGRATE

$$\text{BLAH}(x, y) = \text{CONST.}$$

$$p = \text{BLAH} \quad \text{OR} \quad p = F(\text{BLAH}).$$

EXAMPLE (BOOK, P. 684) FIND GENERAL SOLN:

$$x \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} - 2u = 0 \quad (20.15)$$

SUMMARY OF SOLN: $p = x e^{-y/2}$

$$u(x, y) = h(x, y) f(x e^{-y/2})$$

FOR $h(x, y)$ CAN CHOOSE, E.G., e^y OR x^2 . SO

$$u(x, y) = e^y f(x e^{-y/2})$$

$$\text{OR} = x^2 g(x e^{-y/2})$$

} GENERAL
SOLUTION

$$\text{NOTE THAT } g(p) = f(p)/p^2.$$

↑
IF THIS FACTOR WERE
NOT A FUNCTION OF
 p , WE'D BE WORRIED!

(WHY?)

INHOMOGENEOUS EQUATIONS & PROBLEMS (§20.3.2)

INHOMOGENEOUS EQUATION:

$$A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + C u = R(x,y) \quad **$$

SUPPOSE $u_1(x,y)$ AND $u_2(x,y)$ ARE SOLUTIONS.

(SHOW:) $u_1 + u_2$ IS NOT A SOLUTION!

BUT SUPPOSE $w(x,y)$ IS ANY SOLN. TO THE HOMOGENEOUS PROBLEM:

$$A \frac{\partial w}{\partial x} + B \frac{\partial w}{\partial y} + C w = 0.$$

THEN $u = u_1 + w$ IS A SOLN. TO **
(AND SO IS $u_2 + w$)

E.G.: $w = u_2 - u_1$ WOULD WORK

IF $w(x,y)$ IS THE GENERAL SOLN. TO THE HOMOGENEOUS EQ, AND $u_1(x,y)$ IS ANY SOLUTION TO THE INHOMOGENEOUS EQUATION, THEN

$$u = u_1 + w$$

IS THE GENERAL SOLUTION TO THE INHOMOGENEOUS EQUATION.

IT IS POSSIBLE FOR THE PROBLEM TO BE INHOMOGENEOUS EVEN WHEN THE EQUATION ISN'T.

E.G.

$$A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + C u = 0$$

WITH INHOMOGENEOUS B.C. $u = F(x)$ ON $y=0$

IF u_1 AND u_2 ARE SOLUTIONS TO THE PROBLEM, THE FOLLOWING ARE NOT:

$$u = u_1 + u_2 \rightarrow u(x, 0) = 2F(x)$$

$$u = \pi u_1 \rightarrow u(x, 0) = \pi F(x)$$

ETC.

THE BOUNDARY CONDITIONS ARE CONSIDERED HOMOGENEOUS IF, FOR ANY $u(x, y)$ THAT SATISFIES THEM, $\lambda u(x, y)$ ALSO SATISFIES THEM.

THE PROBLEM IS HOMOGENEOUS IF WE HAVE BOTH

- (i) HOMOGENEOUS EQUATION
- (ii) HOMOGENEOUS B.C.'S

EXAMPLE

SOLVE

INHOMOGENEOUS EQ.

$$y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 3x \quad (20.18)$$

SUBJECT TO $u(1,0) = 2$. ← INHOMOGENEOUS B.C.

① FIRST, SOLVE THE HOMOGENEOUS EQN:

$$y \frac{\partial w}{\partial x} - x \frac{\partial w}{\partial y} = 0$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$\int x dx + \int y dy = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} = C$$

GEN SOLN $w = f(p)$ WHERE $p = x^2 + y^2$

② NOW SEEK THE GENERAL SOLUTION (WITHOUT B.C.'S) TO THE INHOMOGENEOUS EQ. (20.18).

IF WE CAN FIND A PARTICULAR SOLN. $v(x,y)$, THEN THE GENERAL SOLN. WILL BE $u = v + w$.

BY INSPECTION: $v(x,y) = -3y$

GEN SOLN TO (20.18) IS THUS

$$u(x,y) = f(x^2 + y^2) - 3y$$



EXAMPLE, CONT.

③ NOW FOR THE B.C. :

$$u(1,0) = 2$$

$$f(1+0) - 0 = 2$$

$$f(1) = 2$$

IF WE CHOOSE, E.G., $f(p) = 2p$, THEN

$$u(x,y) = \overbrace{2x^2 + 2y^2}^{f(p)} - 3y + g(x^2 + y^2)$$

WHERE g IS ANY FUNCTION WITH $g(1) = 0$.

COULD ALSO CHOOSE $f(p) = 2$! THEN

$$u(x,y) = 2 - 3y + g(x^2 + y^2), \quad g(1) = 0$$

ASSERTION: EACH OF THESE IS THE GENERAL SOLUTION. DO THEY AGREE?

$$2x^2 + 2y^2 - 3x + g_1 = 2 - 3x + g_2$$

$$g_2 = g_1 + \underbrace{2p - 2}$$

THIS IS $g_3(p)$. NOTE THAT $g(1) = 0$.