

§ 20.3.3 SECOND-ORDER EQUATIONS (USING THE SAME TRICKS!) ①

THE MOST GENERAL LINEAR, 2nd ORDER PDE ON 2 VARIABLES WOULD BE

$$A(x,y) \frac{\partial^2 u}{\partial x^2} + B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2} + D(x,y) \frac{\partial u}{\partial x} + E(x,y) \frac{\partial u}{\partial y} + F(x,y)u = R(x,y).$$

WE BEGIN WITH A SIMPLER CASE:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0 \quad \star \quad (20.20)$$

A, B, C CONSTANTS.

THIS CLASS OF EQUATIONS INCLUDES:

- 1d WAVE EQ.
- 2d LAPLACE EQ.

BUT DOES NOT INCLUDE:

- DIFFUSION EQ.
 - SCHRÖDINGER EQ.
- } REQUIRE 1st DERIVATIVES

ASSUME SOLN. OF FORM:

$$u(x,y) = f(p), \quad p = p(x,y).$$

NOW CONSIDER THE DERIVATIVES:

$$\frac{\partial u}{\partial x} = f' \frac{\partial p}{\partial x}, \quad \frac{\partial u}{\partial y} = f' \frac{\partial p}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2} = f'' \left(\frac{\partial p}{\partial x}\right)^2 + f' \frac{\partial^2 p}{\partial x^2}$$
$$\frac{\partial^2 u}{\partial x \partial y} = f'' \frac{\partial p}{\partial x} \frac{\partial p}{\partial y} + f' \frac{\partial^2 p}{\partial x \partial y}$$
$$\frac{\partial^2 u}{\partial y^2} = f'' \left(\frac{\partial p}{\partial y}\right)^2 + f' \frac{\partial^2 p}{\partial y^2}$$

THESE TERMS ARE ANNOYING.

WHEN WE PLUG BACK INTO \star , WE HOPE TO BE ABLE TO ELIMINATE ALL APPEARANCES OF f . THIS SEEMS DOOMED TO FAIL.

SLEAZY TRICK: CHOOSE $p = ax + by$

SO THAT THE 2nd DERIVATIVES GO AWAY. THEN \star BECOMES:

$$A f'' a^2 + B f'' ab + C f'' b^2 = 0$$

$$Aa^2 + Bab + Cb^2 = 0$$

→ NOTE THAT ONLY THE RATIO (b/a) MATTERS. DIVIDE BY a^2 , SO

$$C \left(\frac{b}{a}\right)^2 + B \left(\frac{b}{a}\right) + A = 0$$

$$\frac{b}{a} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C} = \left\{ \begin{matrix} \text{+ ROOT} \\ \lambda_1, \lambda_2 \\ \text{- ROOT} \end{matrix} \right\}$$

SO WE HAVE THE 2 POSSIBILITIES

$$p_1 = x + \lambda_1 y, \quad p_2 = x + \lambda_2 y$$

GENERAL SOLN:

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$$

EXAMPLE : 1D WAVE EQUATION

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

COMPARING WITH Δ , $A=1$, $B=0$, $C = -\frac{1}{c^2}$

$$\therefore \lambda_1 = \frac{\sqrt{1/c^2}}{1/c^2} = c$$

$$\lambda_2 = \frac{-\sqrt{1/c^2}}{1/c^2} = -c$$

SO $u(x, t) = f(x - ct) + g(x + ct)$

WHICH IS WHAT WE SHOULD EXPECT.

EXAMPLE

NOW, TRY LAPLACE'S EQUATION:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$A=C=1, \quad B=0$$

$$\lambda = \frac{b}{a} = \frac{\pm\sqrt{-4}}{2} = \pm i$$

$$u = f(x+iy) + g(x-iy)$$

WHICH IS A RESULT FAMILIAR TO STUDENTS OF COMPLEX ANALYSIS.

NOTE THAT:

BOTH REAL

$$\nabla^2 f(z) = 0, \quad f(z) = \underbrace{\text{Re } f}_{\swarrow} + i \underbrace{\text{Im } f}_{\searrow}$$

$$\Rightarrow \text{Re}(\nabla^2 f(z)) = \text{Im}(\nabla^2 f(z)) = 0$$

$$\Rightarrow \nabla^2(\text{Re } f(z)) = \nabla^2(\text{Im } f(z)) = 0$$

SO EACH $f(x+iy)$ REPRESENTS 2 REAL SOLNS. IN 2D.

ASIDE: CLASSIFICATION OF PDE'S OF FORM *

$B^2 > 4AC \rightarrow$ HYPERBOLIC EQUATIONS

$B^2 < 4AC \rightarrow$ ELLIPTIC "

$B^2 = 4AC \rightarrow$ PARABOLIC "

FOR A, B, C REAL, HYPERBOLIC EQUATIONS
HAVE SOLUTIONS LIKE $f(x + \alpha y)$, $\alpha \in \mathbb{R}$.

ELLIPTIC EQUATIONS HAVE SOLUTIONS LIKE $f(x + i\beta y)$, $\beta \in \mathbb{R}$.

Q: WHAT TYPE IS THE WAVE EQ?

A: HYPERBOLIC

Q: WHAT TYPE IS LAPLACE'S EQ?

A: ELLIPTIC

PARABOLIC EQUATIONS (^{still} § 20.3.3, STARTING P. 690)

FOREGOING APPROACH DOESN'T QUITE WORK FOR
PARABOLIC EQUATIONS ($B^2 = 4AC$); WE WOULD
GET ONLY

$$u(x, y) = f\left(x - \frac{B}{2C}y\right)$$

THIS IS OK, EXCEPT THAT WE

→ SHOULD EXPECT 2 COMBINATIONS, SINCE
IT IS A 2nd ORDER EQUATION.

(7)

TO FIND THE "REST" OF THE SOLN, TRY

$$u(x, y) = h(x, y) g\left(x - \frac{B}{2C} y\right)$$

DERIVATIVES ARE:

$$\frac{\partial u}{\partial x} = \frac{\partial h}{\partial x} g + h g'$$

$$\frac{\partial u}{\partial y} = \frac{\partial h}{\partial y} g - \frac{B}{2C} h g'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 h}{\partial x^2} g + 2 \frac{\partial h}{\partial x} g' + h g''$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 h}{\partial x \partial y} g - \frac{B}{2C} \frac{\partial h}{\partial x} g' + \frac{\partial h}{\partial y} g' - \frac{B}{2C} h g''$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 h}{\partial y^2} g - \frac{B}{C} \frac{\partial h}{\partial y} g' + \frac{B^2}{4C^2} h g''$$

SUB INTO \star :

$$A g \frac{\partial^2 h}{\partial x^2} + 2A g' \frac{\partial h}{\partial x} + A h g''$$

$$+ B g \frac{\partial^2 h}{\partial x \partial y} - \frac{B^2}{2C} \frac{\partial h}{\partial x} g' + B \frac{\partial h}{\partial y} g' - \frac{B^2}{2C} h g''$$

$$+ C g \frac{\partial^2 h}{\partial y^2} - B g' \frac{\partial h}{\partial y} + \frac{B^2}{4C} h g'' = 0$$

THE CASE WE'RE INTERESTED IN IS

$$B^2 - 4AC = 0$$

i.e. $A = B^2/4C$

- $\Rightarrow g''$ TERMS CANCEL
- $\Rightarrow g'$ TERMS CANCEL

WE ARE LEFT WITH (AS USUAL!)

$$\left(A \frac{\partial^2 h}{\partial x^2} + B \frac{\partial^2 h}{\partial x \partial y} + C \frac{\partial^2 h}{\partial y^2} \right) g = 0$$

$\Rightarrow h(x,y)$ MUST BE A PARTICULAR SOLN. TO \star

OUR GENERAL SOLN WILL BE:

$$u(x,y) = f\left(x - \frac{B}{2C}y\right) + h g\left(x - \frac{B}{2C}y\right)$$

\rightarrow IT MAY BE TEMPTING TO CHOOSE $h = x - \frac{B}{2C}y$, BUT THIS WON'T WORK. (WHY?)

INSTEAD CHOOSE $h = x$.

Q: IS $h = x$ A SOLN TO \star ?

A: YES.

Q: IS $h = x$ A SOLN FOR THE ELLIPTIC AND HYPERBOLIC CASES TOO?

A: YES.

Q: IS $h = x$ INCLUDED IN OUR "GENERAL" SOLNS?

A: YES. $h = x = \frac{1}{2}(P_1 + P_2)$, E.G., FOR ELLIPTIC CASE.

RESULT:

FOR $B^2 = 4AC$, (PARABOLIC)

$$u(x,y) = f\left(x - \frac{B}{2C}y\right) + x g\left(x - \frac{B}{2C}y\right)$$

EXAMPLE (FROM BOOK)

$$\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\text{B.C.: } \begin{aligned} u(0,y) &= 0 \\ u(x,1) &= x^2 \end{aligned}$$

$$\text{SOLN: } \begin{aligned} &\text{- PARABOLIC, } p = x - y \\ &\left. \begin{aligned} &\text{- } f(p) = 0 \\ &\text{- } g(p) = p + 1 \end{aligned} \right\} \text{BC CONSEQUENCES} \end{aligned}$$

$$u(x,y) = x(p+1) = x^2 - xy + x.$$