

TODAY: INITIAL VALUE PROBLEMS

§ 20.4 WAVE EQUATION (SOLUTION OF D'ALEMBERT)

§ 20.5 DIFFUSION EQUATION (BOOK DROPPED THE BALL,
I'LL MAKE IT WORK!)

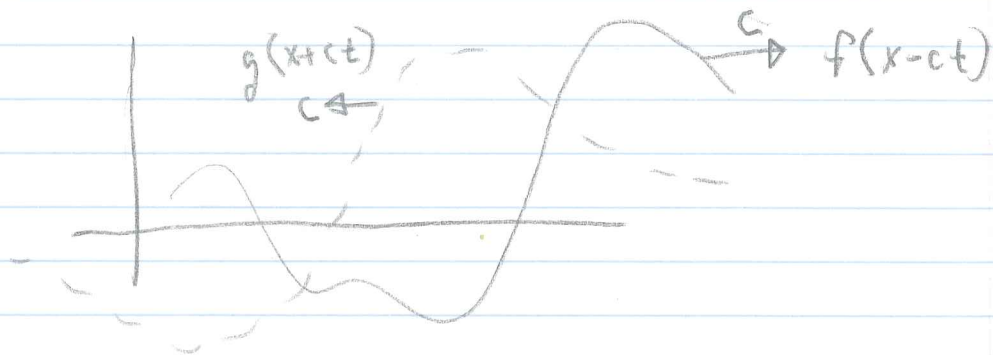
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§ 20.4 THE WAVE EQUATION, AND D'ALEMBERT'S SOLUTION

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

GEN SOLN: $u(x, t) = f(x-ct) + g(x+ct)$

WE HAVE ALREADY DISCUSSED THE PHYSICAL INTERPRETATION OF THIS SOLUTION:



D'ALEMBERT'S SOLUTION TO THE INITIAL VALUE PROBLEM:

SUPPOSE WE KNOW u AND $\frac{\partial u}{\partial t}$ AT TIME $t=0$:

$$u(x, 0) = \phi(x),$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x).$$

Q: IS THIS A COMPLETE SET OF INITIAL CONDITIONS?

(I.E. DO THESE B.C.'S IMPLY A UNIQUE SOLN?)

ASIDE
(NOT IN TEXT)

"THE IDEA BEHIND INITIAL VALUE PROBLEMS!"

GIVEN: EQUATION OF FORM

$$\frac{\partial}{\partial t} \vec{s}(\vec{r}, t) = \vec{F}(\vec{s}(\vec{r}, t), \left. \frac{\partial \vec{s}}{\partial r_i} \right|_{t_0}, \vec{r}, t) *$$

PROVE: THEN IF AT TIME t_0 , WE KNOW \vec{s} , ALL FUTURE VALUES OF \vec{s} ARE KNOWN.

PROOF BY CONSTRUCTION:

SUPPOSE WE KNOW \vec{s} AT TIME t_0 , $\vec{s}(\vec{r}, t_0)$.

$$\text{THEN } \frac{\partial \vec{s}}{\partial t} = \frac{\vec{s}(\vec{r}, t_0 + dt) - \vec{s}(\vec{r}, t_0)}{dt} = \vec{F}$$

$$\vec{s}(\vec{r}, t_0 + dt) = \vec{s}(\vec{r}, t_0) + \vec{F}(\vec{s}(\vec{r}, t_0), \left. \frac{\partial \vec{s}}{\partial r_i} \right|_{t_0}, \vec{r}, t_0) dt$$

THUS, KNOWLEDGE OF $\vec{s}(\vec{r}, t_0)$ IMPLIES KNOWLEDGE OF \vec{s} A MOMENT dt LATER. THIS IN TURN IMPLIES KNOWLEDGE OF \vec{s} FOR ALL FUTURE TIMES.

→ THE ACT OF PUTTING A DIFFERENTIAL EQUATION IN THIS FORM ALLOWS US TO IDENTIFY \vec{s} , THE "STATE" OF THE SYSTEM, WHICH IS EXACTLY THE INFORMATION NEEDED TO DETERMINE THE FUTURE BEHAVIOR OF THE SYSTEM.

* ESSENTIALLY, \vec{F} IS A FUNCTION OF HOW THINGS ARE AT TIME t_0 .

EXERCISE: PUT THE WAVE EQUATION INTO THE FORM

$$\frac{\partial}{\partial t} \vec{s}(\vec{r}, t) = \vec{F}(\vec{s}, \vec{r}, t)$$

THE WAVE EQ.,

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial x^2},$$

Q: IS SECOND-ORDER IN TIME. IS OUR TASK THEN HOPELESS?

A: NO, BECAUSE 2nd ORDER EQUATION CAN ALWAYS BE WRITTEN AS 2 COUPLED 1st ORDER EQUATIONS.

TRICK! NEW VARIABLE $\dot{u} \equiv \frac{\partial u}{\partial t}$

$$\frac{\partial \dot{u}}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \dot{u}$$

$$\frac{\partial}{\partial t} \begin{bmatrix} \dot{u} \\ u \end{bmatrix} = \begin{bmatrix} c^2 \frac{\partial^2 u}{\partial x^2} \\ \dot{u} \end{bmatrix} \quad \checkmark$$

$$\vec{s} = \begin{bmatrix} \dot{u} \\ u \end{bmatrix}$$

← COMPLETE DESCRIPTION OF STATE OF THE SYSTEM AT TIME t

SO YES

$$\left. \begin{aligned} u(x, 0) &= \phi(x) \\ \partial_t u(x, 0) &= \psi(x) \end{aligned} \right\}$$

THESE ARE JUST WHAT WE NEED!

BACK TO D'ALEMBERT'S SOLN:

$$u(x,t) = f(x-ct) + g(x+ct)$$

FIND f & g THAT SATISFY BC'S:

$$\phi(x) = u(x,0) = f(x) + g(x)$$

$$\psi(x) = \partial_t u(x,0) = -c f'(x) + c g'(x)$$

$$\frac{1}{c} \int_{p_0}^p \psi(q) dq = \int_{p_0}^p [-f'(q) + g'(q)] dq$$

$$\frac{1}{c} \int_{p_0}^p \psi(q) dq = -f(p) + g(p) + f(p_0) - g(p_0)$$

ADD
EQ'S

$$\phi(p) = f(p) + g(p)$$

$$g(p) = \frac{1}{2} [g(p_0) - f(p_0)] + \frac{1}{2} \phi(p) + \frac{1}{2c} \int_{p_0}^p \psi(q) dq; \quad (20.30)$$

SIMILARLY, IF WE SUBTRACT THE EQ'S,

$$f(p) = -\frac{1}{2} [g(p_0) - f(p_0)] + \frac{1}{2} \phi(p) - \frac{1}{2c} \int_{p_0}^p \psi(q) dq. \quad (20.29)$$

NOW, A WONDERFUL THING HAPPENS WHEN WE PLUG THESE RESULTS BACK INTO THE GENERAL SOLN...

$$u(x,t) = f(x-ct) + g(x+ct)$$

IF THE SAME p_0 IS USED FOR BOTH 20.29 AND 20.30,

$$u(x,t) = \frac{1}{2} \phi(x-ct) - \frac{1}{2c} \int_{p_0}^{x-ct} \psi(\xi) d\xi \\ + \frac{1}{2} \phi(x+ct) + \frac{1}{2c} \int_{p_0}^{x+ct} \psi(\xi) d\xi$$

NOW, LET $-\int_{p_0}^{x-ct} \rightarrow + \int_{x-ct}^{p_0}$, SO

$$u(x,t) = \frac{1}{2} \left[\phi(x-ct) + \phi(x+ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

(20.31)

p_0 HAS GONE AWAY!

IT IS FUN TO TRY AND DEVELOP PHYSICAL INTUITION FOR THIS SOLUTION...

FIRST TERM: SUPERPOSITION OF RIGHT- AND LEFT-GOING PIECES OF $\phi(x) = u(x,0)$.

SECOND TERM: EACH $\dot{u}(x',0)$ WITHIN $\pm ct$ CONTRIBUTES $\dot{u}(x',0) dt' = \dot{u}(x',0) dx'/c$ TO $u(x,t)$ AS IT PROPAGATES BY TO THE RIGHT AND LEFT.

§20.5 DIFFUSION EQUATION

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u(x,t)}{\partial x^2} \quad \star\star$$

Q: BASED ON OUR DISCUSSION OF INITIAL VALUE PROBLEMS, WHAT BOUNDARY CONDITIONS WILL WE NEED AT TIME $t=0$?

A: THE PDE IS FIRST-ORDER IN TIME, ONLY $u(x,0)$ IS NEEDED.

OUR USUAL METHOD WILL NOT WORK QUITE THE WAY IT HAS IN THE PAST. (IF $u = h f(\eta)$, CAN'T GET RID OF f')

NOTE THAT $\eta = \frac{x^2}{kt}$ IS DIMENSIONLESS. LOOK FOR

A SOLUTION OF THE FORM

$$u(x,t) = f(\eta)$$

DERIV'S: $\frac{\partial u}{\partial x} = \frac{2x}{kt} f'$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2}{kt} f' + \left(\frac{2x}{kt}\right)^2 f''$$

$$\frac{\partial u}{\partial t} = -\frac{x^2}{kt^2}$$

SUB INTO $\star\star$:

REWRITE
PP 7-8

$$4\eta f'' + (2+\eta) f' = 0$$

THIS IS AN ODE.

$$\int d\eta \left(\frac{f''}{f'} \right) = \int \left(-\frac{1}{2\eta} - \frac{1}{4} \right) d\eta$$

$$\ln f' = -\frac{1}{2} \ln \eta - \frac{\eta}{4} + C$$

$$\ln(\eta^{1/2} f') = -\frac{\eta}{4} + C$$

$$f'(\eta) = A \eta^{-1/2} e^{-\eta/4}$$

$$u(x,t) = f(\eta) = A \int_{\eta_0}^{\eta} \eta^{-1/2} e^{-\eta/4} d\eta \quad \leftarrow \text{DUMMY VAR}$$

$$\text{LET } \xi = \frac{\eta^{1/2}}{2} = \frac{x}{2(kt)^{1/2}},$$

$$d\xi = \frac{1}{4} \eta^{-1/2} d\eta$$

$$\text{THEN } u(x,t) = B \int_{\xi_0}^{\xi} e^{-v^2} dv \quad \leftarrow \text{NEW DUMMY VAR}$$

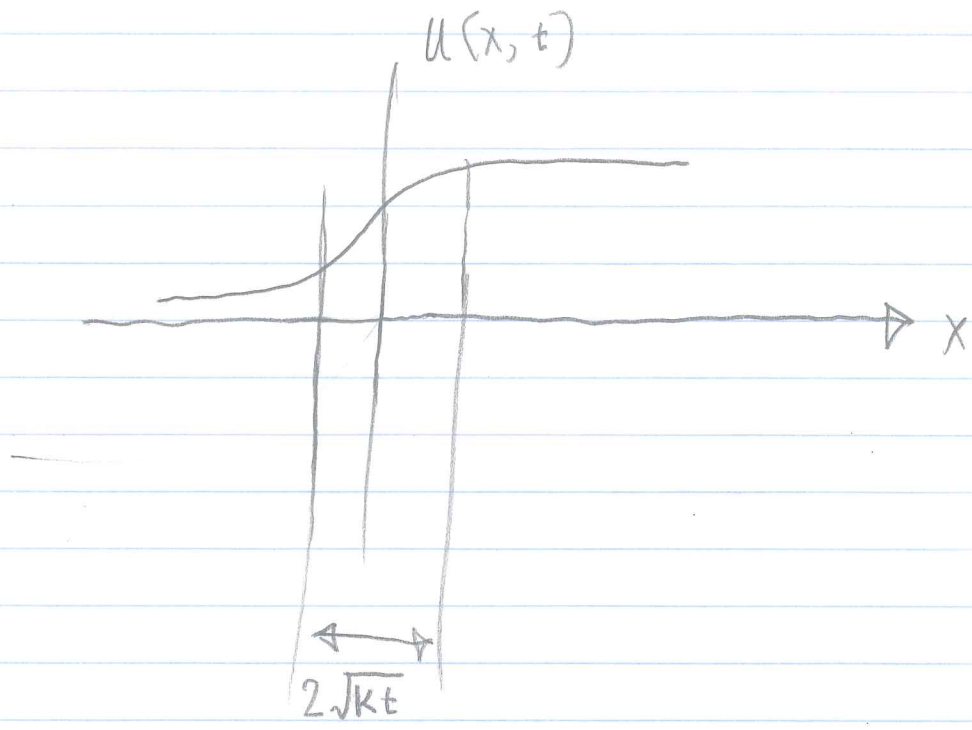
TO UNDERSTAND THIS SOLN, LET $\xi_0 \rightarrow -\infty$

$$u(x,t) = B \int_{-\infty}^{x/2\sqrt{kt}} e^{-v^2} dv \quad \star\star\star$$

THIS IS
WHERE
RHS
STOPS.

Q: THIS IS JUST A PARTICULAR SOLN. IS IT ANY USE?

THIS FUNCTION LOOKS LIKE:



- INTERPRET PHYSICALLY IN TERMS OF DIFFUSION
- IN LIMIT $x \rightarrow \infty$, CHOOSE B SO THAT $u(x, t) \rightarrow 1$
- THEN, IN LIMIT $t \downarrow 0$,

$$u(x, 0) \rightarrow H(x).$$

THIS IS USEFUL! CAN BUILD ANYTHING OUT OF HEAVISIDE FUNCTIONS!

Now, A TRICK WITH SUPERPOSITION:

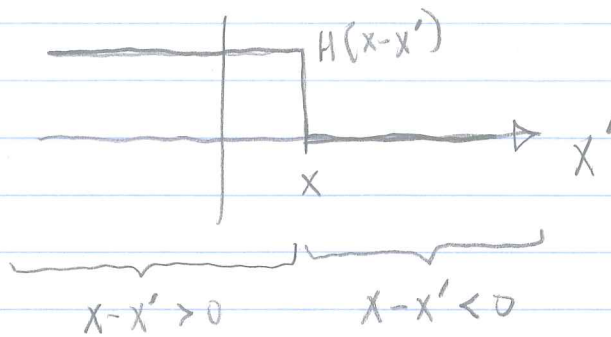
FUNDAMENTAL THM OF CALCULUS:

$$\underbrace{u(x,0) = \varphi(x)}_{\text{INITIAL CONDITION}} = \int_{-\infty}^x \varphi'(x') dx' + \varphi(-\infty)$$

LET THIS BE ZERO

$$= \int_{-\infty}^{\infty} H(x-x') \varphi'(x') dx' \quad \odot$$

BECAUSE:



$u(x,0)$ IS THUS A SUPERPOSITION OF HEAVISIDE FUNCTIONS.

OUR PDE IS INVARIANT UNDER $x \rightarrow x-x'$ THUS SO IS OUR SOLN $\star\star\star$:

$$Q(x-x_0, t) = B \int_{-\infty}^{(x-x_0)/2\sqrt{kt}} e^{-v^2} dv$$

SINCE $u(x,t) = Q(x-x', t)$ IS A SOLN TO $\star\star$, SO IS

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} Q(x-x', t) \varphi'(x') dx'}$$

SINCE $Q \rightarrow H(x-x')$ AT $t \rightarrow 0$, THE ABOVE SOLN (BY COMPARISON WITH \odot) SATISFIES OUR INITIAL CONDITIONS!