

§ 20.6.1 — CHARACTERISTICS FOR 1ST ORDER EQ'S

CONSIDER THE GENERAL 1ST ORDER PDE ON 2 VAR'S:

$$A(x,y) \frac{\partial u}{\partial x} + B(x,y) \frac{\partial u}{\partial y} = F(x,y,u) \quad \star$$

NOTE HOW u HAS BEEN MOVED TO THE RHS.

Q: WHAT KIND OF BOUNDARY CONDITIONS HAVE, IN GENERAL, BEEN SUFFICIENT TO SPECIFY A PARTICULAR SOLN? E.G.:

(A) $u(x,0) = \varphi(x)$.

(B) $u(1,3) = 4$.

A: (A) YES, (B) NO.

LET'S TRY TO MAKE UP A MORE GENERALIZED VERSION OF (A):

$$\text{LET } u(x,y) = \varphi(s)$$

ON PARAMETRIC CURVE

$$x = x(s), \quad y = y(s).$$

NOW, CONSIDER

$$\frac{du}{ds} = \frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds} = \frac{d\varphi}{ds} \quad \star\star$$

UNKNOWN

KNOWN



WE CAN REWRITE * AND ** AS:

$$\begin{pmatrix} A(x,y) & B(x,y) \\ \frac{dx}{ds} & \frac{dy}{ds} \end{pmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} F(x,y,u) \\ \frac{d\phi}{ds} \end{bmatrix}$$

THIS CAN BE SOLVED USING CRAMER'S RULE (P. 299) TO GIVE US $\partial_x u$ AND $\partial_y u$ ON THE BOUNDARY CURVE $(x(s), y(s))$. (WE WON'T DO THIS.)

NOTICE:

CRAMER'S RULE HAS THE DETERMINANT OF THE MATRIX IN THE DENOMINATOR. THERE IS NO SOLN IF

$$\begin{vmatrix} A & B \\ x' & y' \end{vmatrix} = 0.$$

SO THE SOLUTIONS TO THE ABOVE EQUATION ARE BOUNDARY CURVES THAT DON'T WORK:

$$A \frac{dy}{ds} - B \frac{dx}{ds} = 0$$

OR $\boxed{\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}} \quad (20.41)$

THIS SHOULD LOOK FAMILIAR — IT IS THE EQUATION FOR SURFACES OF CONSTANT $p(x,y)$,

WHERE $u(x,y) = \overset{\text{POSSIBLY 1}}{h(x,y)} f(p).$

SOLUTIONS TO 20.41 LOOK LIKE

$$p(x,y) = \text{CONST.}$$

THESE CURVES OF CONSTANT p ARE CALLED CHARACTERISTICS.

IMPLICATIONS:

- SETTING THE VALUE OF $u(x,y)$ AT ONE POINT ON A CHARACTERISTIC CURVE DETERMINES u EVERYWHERE ALONG THAT CURVE, BECAUSE IT FIXES $f(p)$ FOR THAT PARTICULAR VALUE OF p .
- IF THE BOUNDARY CURVE RUNS TANGENT TO THE CHARACTERISTICS AT ANY POINT, THEN u IS OVERDETERMINED BECAUSE IT FIXES $f(p)$ FOR A CERTAIN p MORE THAN ONCE.
- SIMILARLY, IF THE BOUNDARY CROSSES A CHARACTERISTIC MORE THAN ONCE, u MAY BE OVERDETERMINED.

INTERPRETATION:

THE CHARACTERISTICS MAP THE PATHS ALONG WHICH INFORMATION FLOWS FROM A BOUNDARY CONDITION.

EXAMPLE (THE BOOK GIVES A DIFFERENT, EQUALLY GOOD EXAMPLE)

1D ADVECTION EQUATION:

$$\frac{\partial u}{\partial t} = -v(x,t) \frac{\partial u}{\partial x} \quad \left(\begin{array}{l} \text{DISCUSS} \\ \text{PHYSICALLY} \end{array} \right)$$

FOR SIMPLICITY, TAKE $v = \text{CONST.}$

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0$$

ASSUME GEN SOLN $u(x,y) = f(p(x,y))$.

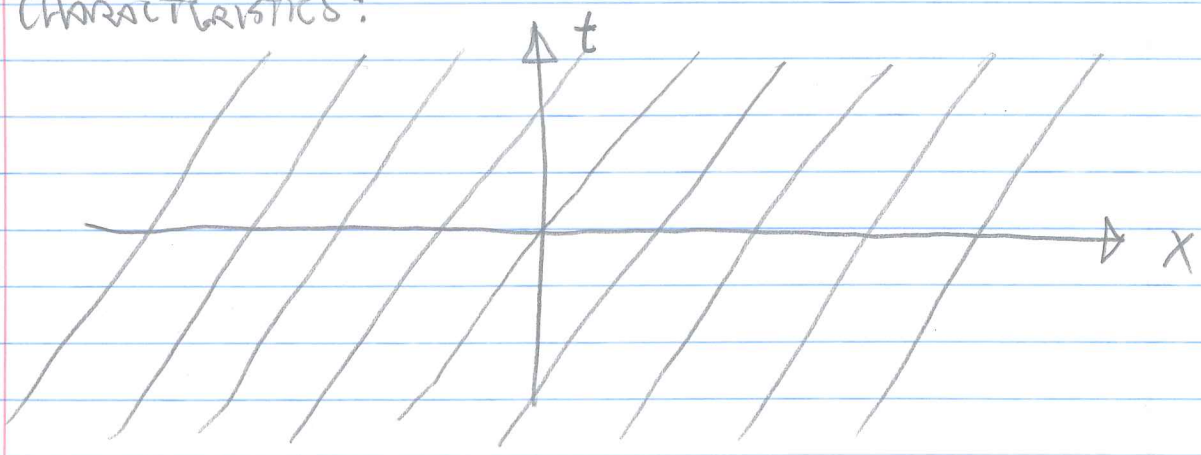
FIND p :

$$\frac{dt}{1} = \frac{dx}{v}$$

$$t = \frac{x}{v} + \text{CONST.}$$

$$p = x - vt \quad ; \quad u = f(x - vt)$$

CHARACTERISTICS:



LINES OF SLOPE $\frac{dt}{dx} = \frac{1}{v}$.



CASE I

SUPPOSE WE HAVE THE BOUNDARY CONDITION

$$u(x, 0) = e^{-x^2}$$

- BOUNDARY CURVE $t=0$
- BC INTERSECTS ALL CHARACTERISTIC CURVES,
- BC INTERSECTS EACH CHARACTERISTIC ONCE.

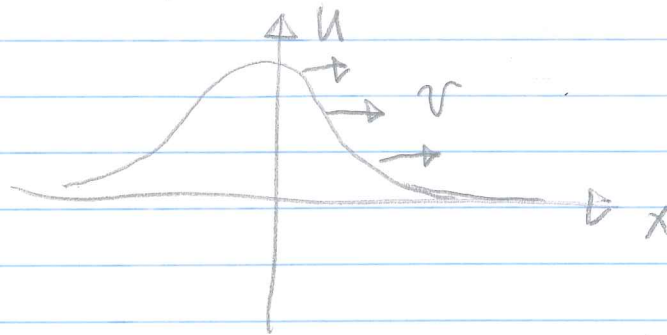
THUS, $u(x, t)$ SHOULD BE FULLY DETERMINED EVERYWHERE IN THE PLANE.

SOLUTION:

$$f(x - vt) = e^{-x^2}$$

$$f(x) = e^{-x^2}$$

$$\text{THUS } u(x, t) = f(x - vt) = e^{-(x - vt)^2}$$



FOR EACH OF THE FOLLOWING CASES,

- (a) SKETCH THE BOUNDARY CURVE
- (b) SPECIFY WHERE $u(x, t)$ IS KNOWN
(AND GIVE THE EXPLICIT SOLUTION)
- (c) SPECIFY WHERE $u(x, t)$ IS UNKNOWN
- (d) SPECIFY WHERE $u(x, t)$ IS OVERDETERMINED

CASE II: $u(x, 0) = \cos x$ FOR $-\pi/2 < x < \pi/2$
 (I DO THIS ONE)

CASE III: $u(0, t) = u(1, t) = x$

CASE IV: $u(x, t) = 1$ ON $x^2 + y^2 = 1$.

CASE V: $u(x, 1 + x/\nu) = x^2$

LET THEM HANDLE THESE.

CASE II:

