

§20.6.2 CHARACTERISTICS FOR 2nd ORDER EQ'S

GENERAL LINEAR 2nd ORDER PDE ON x, y :

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} = F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \quad (*)$$

EVERYTHING BUT ∂^2 MOVED
TO THE RHS.

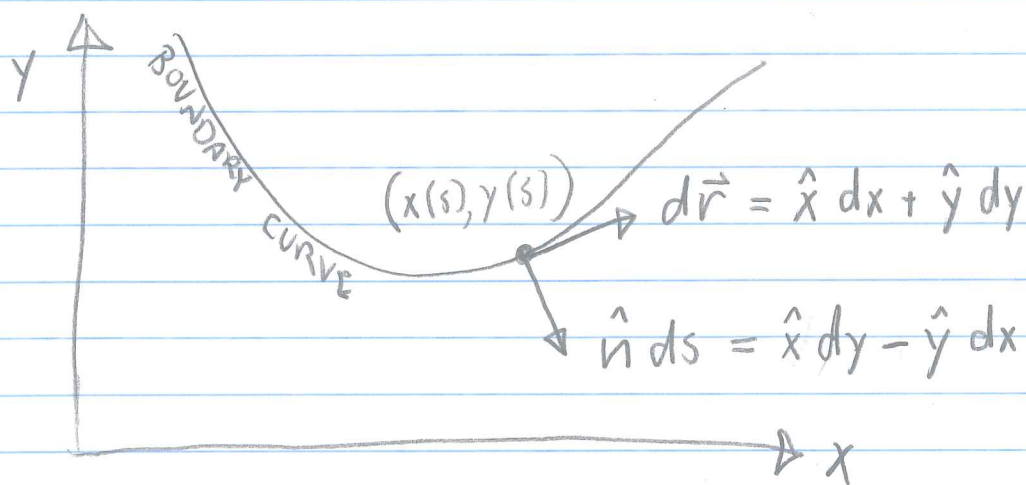
TYPES OF BC.'S :

DIRICHLET $u(x, y) = \varphi(s)$

NEUMANN $\hat{n} \cdot \nabla u = \psi(s)$

WHERE \hat{n} IS NORMAL TO THE BOUNDARY

CAUCHY $\varphi(s)$ AND $\psi(s)$ BOTH SPECIFIED.



WE'LL DO JUST CAUCHY
 NOW, REWRITE B.C.'S AS FOLLOWS

(8)

$$\frac{\partial u}{\partial n} = (\hat{n} \cdot \nabla) u = \left(\frac{\hat{n} ds}{ds} \cdot \nabla \right) u$$

$$= \left[\frac{\hat{x} dy - \hat{y} dx}{ds} \cdot \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \right) \right] u$$

$$= \left(\frac{dy}{ds} \frac{\partial u}{\partial x} - \frac{dx}{ds} \frac{\partial u}{\partial y} \right) = \psi(s) \quad (1)$$

SIMILARLY

$$\frac{\partial u}{\partial s} = \left(\frac{d\vec{r}}{ds} \cdot \nabla \right) u$$

$$= \left[\frac{\hat{x} dx + \hat{y} dy}{ds} \cdot \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \right) \right] u$$

$$= \left(\frac{dx}{ds} \frac{\partial u}{\partial x} + \frac{dy}{ds} \frac{\partial u}{\partial y} \right) = \frac{d\phi}{ds} \quad (2)$$

SOLVE THE CIRCLED EQ'S FOR $\frac{\partial u}{\partial x}$ AND $\frac{\partial u}{\partial y}$:

$$\frac{\partial u}{\partial x} = \frac{\frac{dy}{ds} \psi(s) + \frac{dx}{ds} \frac{d\phi}{ds}}{\left[\left(\frac{dy}{ds} \right)^2 + \left(\frac{dx}{ds} \right)^2 \right]} = G(s) \quad (\text{KNOWN})$$

← 2x2 MATRIX DETERMINANT ALWAYS NONZERO, SO SOLN. ALWAYS EXISTS.

SIMILARLY,

$$\frac{\partial u}{\partial y} = H(s) \quad (\text{KNOWN})$$

ALSO IS GUARANTEED TO EXIST.

NOW, TAKE d/ds OF $G(s)$ AND $H(s)$.

NOTE THAT

$$\frac{d}{ds} = \frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dy}{ds} \frac{\partial}{\partial y},$$

SO

$$\frac{dG}{ds} = \frac{dx}{ds} \frac{\partial^2 u}{\partial x^2} + \frac{dy}{ds} \frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{dH}{ds} = \frac{dx}{ds} \frac{\partial^2 u}{\partial x \partial y} + \frac{dy}{ds} \frac{\partial^2 u}{\partial y^2}$$

COMBINE THESE WITH (*) TO FORM A 3×3 MATRIX EQ. FOR THE 2nd PARTIALS OF u ALONG THE BOUNDARY CURVE:

$$\begin{pmatrix} A(x,y) & B(x,y) & C(x,y) \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \end{pmatrix} \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} \end{bmatrix} = \begin{bmatrix} F \\ \frac{dG}{ds} \\ \frac{dH}{ds} \end{bmatrix}$$

AGAIN, WE FIND THAT OUR B.C.'S ARE OVERDETERMINED (I.E. THERE IS NO SOLN) IF THE MATRIX DETERMINANT IS ZERO.

CONSEQUENTLY, THE CHARACTERISTICS ARE GIVEN BY

$$|M| = 0 \dots$$

$$\begin{vmatrix} A & B & C \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \end{vmatrix} = 0$$

$$A \left(\frac{dy}{ds}\right)^2 - B \left(\frac{dx}{ds}\right) \left(\frac{dy}{ds}\right) + C \left(\frac{dx}{ds}\right)^2 = 0$$

MULTIPLY BY $\left(\frac{ds}{dx}\right)^2$:

$$A \left(\frac{dy}{dx}\right)^2 - B \left(\frac{dy}{dx}\right) + C = 0$$

FOR ONCE, WE GET TO SOLVE A NONLINEAR ODE! ;)

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

IF WE CAN INTEGRATE THIS LINEAR 1ST ORDER EQUATION, WE WILL FIND TWO FAMILIES OF CHARACTERISTICS, ONE FOR EACH ROOT.

THIS WILL WORK BEST FOR HYPERBOLIC EQUATIONS ($B^2 - 4AC > 0$). RHB GIVES RESULTS BUT NO DERIVATIONS FOR THE OTHER KINDS.

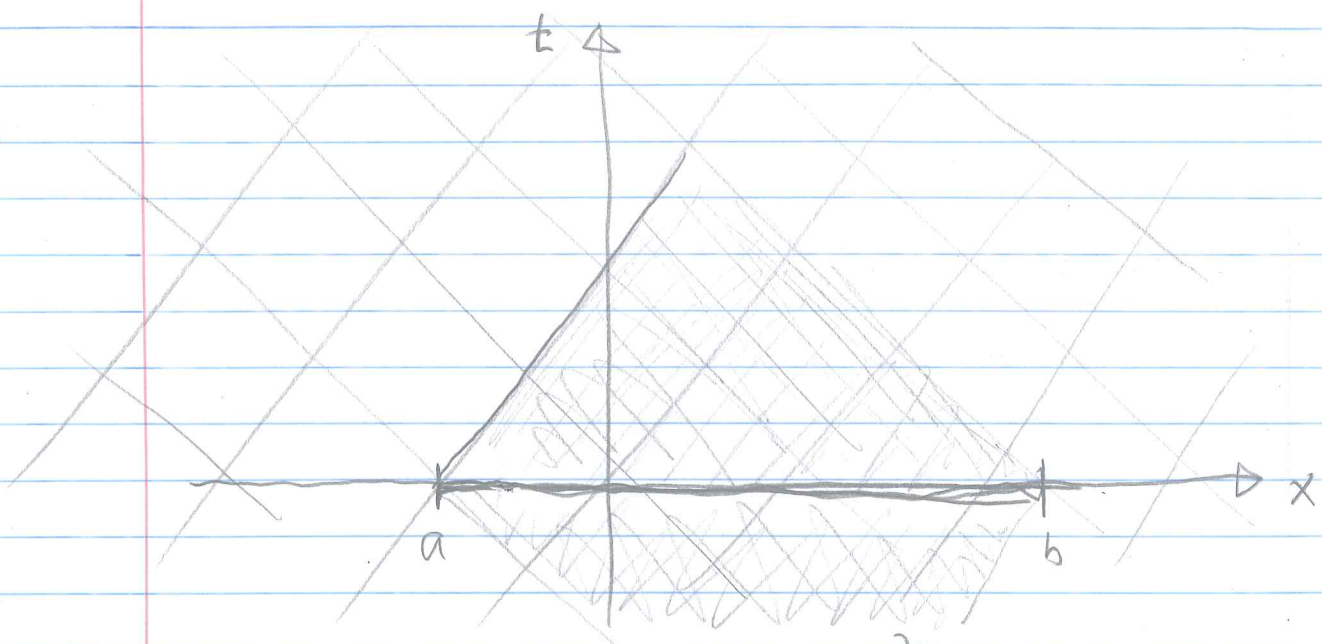
EXAMPLE: 1D WAVE EQ (AS IN RILEY)

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

CHARACTERISTICS:

$$\frac{dx}{dt} = \pm c$$

KNOWING $u(x,t)$ REQUIRES INFORMATION THAT PROPAGATES ALONG BOTH SETS OF CURVES:



E.g., IF WE KNOW u AND $\frac{\partial u}{\partial t}$ AT $t=0$ OVER SOME INTERVAL $a < x < b$, THEN $u(x,t)$ IS KNOWN ONLY OVER THE SHADED DOMAIN. RECALL: D'ALEMBERT'S SOLN:

GIVEN $u(x,0) = \phi(x)$ AND $\partial_t u(x,0) = \psi(x)$,

$$u(x,t) = \frac{1}{2} [\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\xi) d\xi$$

DISCUSS. 2nd TERM SHOWS THAT OUR THINKING MAY BE SLIGHTLY NAIVE!