

CH. 21 [SEPARATION OF VARIABLES, §21.1]

EXAMPLE

LET'S WORK BY EXAMPLE. SUPPOSE WE HAVE LAPLACE'S EQUATION (RHS USES THE WAVE EQUATION).

$$\nabla^2 u(\vec{r}) = 0 \quad \star$$

SEEK A "SEPARABLE" SOLUTION, MEANING:

$$u(\vec{r}) = X(x) Y(y) Z(z).$$

\star BECOMES:

$$(\partial_x^2 + \partial_y^2 + \partial_z^2) (X Y Z) = 0.$$

$$X'' Y Z + X Y'' Z + X Y Z'' = 0$$

DIVIDE BY $u(\vec{r}) = X Y Z$:

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

↑ ↑ ↑
FUNCTION OF / ONLY Z ONLY
X ONLY

CRUCIAL POINT:

$$\text{IF } f(x) + g(y) + h(z) = 0,$$

THEN $\left. \begin{matrix} f(x) = A \\ g(y) = B \\ h(z) = C \end{matrix} \right\}$ ALL CONSTANTS $\in \mathbb{C}$

AND $A + B + C = 0.$

THIS GIVES US 3 ODES:

CHOSEN NAME FOR OUR CONSTANT. $\alpha \in \mathbb{C}$.

$$\frac{X''}{X} = -\alpha^2 \Rightarrow X'' = -\alpha^2 X$$

SIMILARLY, $Y'' = -\beta^2 Y$

$$Z'' = -\gamma^2 Z$$

SOLUTIONS:

$$X = A e^{i\alpha x} + B e^{-i\alpha x}$$

$$Y = C e^{i\beta y} + D e^{-i\beta y}$$

$$Z = E e^{i\gamma z} + F e^{-i\gamma z}$$

WHERE $\alpha^2 + \beta^2 + \gamma^2 = 0$

NOTE THAT α, β, γ CANNOT ALL BE $\in \mathbb{R}$!

CONSIDER, E.G., BOUNDARY CONDITIONS ON $0 < x < \infty$:

$$\left. \begin{aligned} u(0, y, z) &= \sin ky \\ u(\infty, y, z) &= 0 \end{aligned} \right\} \text{CLOSED, DIRICHLET}$$

Q: CAN YOU GUESS THE SOLN?

A: $u(x, y, z) = e^{-kx} \sin ky$

EX

Q: NOW YOU TRY ONE:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad **$$

FIND THE MOST GENERAL SEPARABLE SOLN.

A: SEP OF VARIABLES:

$$u = X(x) T(t)$$

$$** \rightarrow X T' = k X'' T$$

$$\frac{1}{k} \frac{T'}{T} = \frac{X''}{X} = -\alpha^2, \alpha \in \mathbb{C}$$

$T = e^{-\alpha^2 k t} \quad \alpha \in \mathbb{C}$
$X = A e^{+i\alpha x} + B e^{-i\alpha x}$

SAVE.

$$\text{OR} = A \sin \alpha x + B \cos \alpha x.$$

Q: WHAT IS THE SOLN. TO THE INITIAL VALUE PROBLEM

$$u(x, 0) = u_1 \sin kx, \text{ FOR } k \frac{\partial^2}{\partial x^2} u(x, t) = \frac{\partial}{\partial t} u(x, t) ?$$

A: $u(x, t) = u_1 \sin kx e^{-k^2 kt}$

Q: HOW ABOUT

$$u(x, 0) = u_0 + u_1 \cos kx ?$$

A: $u(x, t) = u_0 + u_1 \cos kx e^{-k^2 kt}$

Q: THEN HOW ABOUT

$$u(x, 0) = \varphi(x) ?$$

HINT: START WITH

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(k') e^{ik'x} dk'$$

AND E.G. $\tilde{\varphi}(k') = \delta(k - k')$

A: IF $\tilde{\varphi}(k') = \delta(k - k')$, THEN

$$u_k(x, 0) = \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

$$\Rightarrow u_k(x, t) = \frac{1}{\sqrt{2\pi}} e^{-k^2 kt} e^{ikx}$$



IN GENERAL, THEN,

$$\sum_k \tilde{T}_k$$

$$u(x,t) = \int_{-\infty}^{\infty} u_k(x,t) \tilde{\varphi}(k) dk$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 kt} e^{ikx} \tilde{\varphi}(k) dk$$

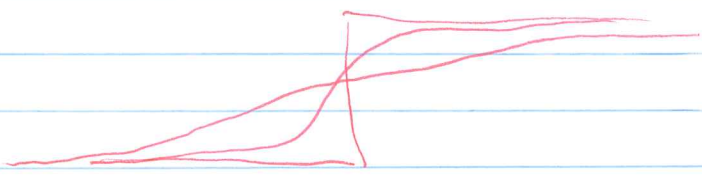
$$u(x,t) = \mathcal{F}_x^{-1} [e^{-k^2 kt} \tilde{\varphi}(k)]$$

(i) SHOW THAT $u(x,t)$ SATISFIES ~~***~~.

(ii) SHOW THAT $u(x,t)$ SATISFIES B.C.



$$u(x,t) \sim e^{-x^2/kt} * \varphi(x)$$



$$t \propto \sigma^2$$

WHAT DID WE DO HERE?

SUPERPOSITION OF SOLUTIONS (§ 21.2):

SUPPOSE WE CAN FIND C_{α_i} SUCH THAT:

$$U(x,t) \Big|_b = \sum_{\alpha_i} C_{\alpha_i} X_{\alpha_i} T_{\alpha_i} \Big|_b = \varphi(x)$$

↑ EVAL. ON BOUNDARY
↑ BOUNDARY VAL'S

SUM OVER SEPARATION CONSTANTS.

THEN THE SOLN. TO THE BOUNDARY VALUE PROBLEM IS JUST:

$$U(x,t) = \sum_{\alpha_i} C_{\alpha_i} X_{\alpha_i} T_{\alpha_i}$$

E.G. INITIAL VALUE PROBLEM

$$U(x,0) = \varphi(x)$$

NOW, SUPPOSE WE ARE FORTUNATE ENOUGH TO HAVE

$$\langle X_{\alpha_i} | X_{\alpha_j} \rangle = \delta_{ij}$$

AND THE X_{α_i} ARE COMPLETE. THEN

IS THIS LIKELY? CONSIDER THAT X_{α_i}, α_i ARE USUALLY SOLN'S TO AN EIGENVALUE PROBLEM.

$$|\varphi\rangle = \sum_{\alpha_i} |X_{\alpha_i}\rangle \langle X_{\alpha_i} | \varphi \rangle$$

MORE GENERALLY, MIGHT NEED A $\rho(x)$ HERE.

$$\varphi(x) = \sum_{\alpha_i} \left[X_{\alpha_i}(x) \int_0^b X_{\alpha_i}(x') \varphi(x') dx' \right]$$

WLOG, LET $T_{\alpha_i}(0) = 1$. THEN

$$\varphi(x) = U(x,0) = \sum_{\alpha_i} X_{\alpha_i}(x) T_{\alpha_i}(0) \int_0^b X_{\alpha_i}(x') \varphi(x') dx'$$

$$\Rightarrow U(x,t) = \sum_{\alpha_i} X_{\alpha_i}(x) T_{\alpha_i}(t) \langle X_{\alpha_i} | \varphi \rangle$$

$C_{\alpha_i} = \langle X_{\alpha_i} | \varphi \rangle$