

BOUNDARY VALUE PROBLEM:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} ; u(0, t) = f(t)$$

FIND $u(x, t)$ FOR $x > 0$

PHYSICAL PICTURE: u = TEMPERATURE OF A SEMI-INFINITE ROD, TEMPERATURE SET AT $f(t)$ AT THE END.

Q: IS B.C. SUFFICIENT? WE WILL SEE...

LET $u = X(x) T(t)$

YES, WE DID THIS BEFORE, BUT NOW WE WANT $T(t)$ TO BE A SET OF ORTHOGONAL FUNCTIONS FROM WHICH WE CAN BUILD $f(t)$.

$$\frac{X T'}{X T} = k \frac{X'' T}{X T}$$

$$\frac{T'}{T} = k \frac{X''}{X} = i\omega \quad (\text{WHY?})$$

$$T = e^{i\omega t}$$

IF $X = e^{\beta x}$, THEN

$$k\beta^2 = i\omega$$

$$\beta = \pm \sqrt{\frac{i\omega}{k}}$$

$$(\text{RECALL } \sqrt{i} = \frac{1}{\sqrt{2}}(1+i))$$

$$\therefore \beta = \pm \sqrt{\frac{\omega}{2k}}(1+i)$$

$$X = A e^{\sqrt{\frac{\omega}{2k}}(1+i)x} + B e^{-\sqrt{\frac{\omega}{2k}}(1+i)x}$$

- ARE OUR BOUNDARY CONDITIONS SUFFICIENT TO CHOOSE A AND B? DISCUSS.
- IS THIS SOLUTION $u = X T$ EQUIVALENT TO WHAT WE HAD BEFORE? CONSIDER $\alpha \in \mathbb{C} \dots$

IN VIEW OF OUR PHYSICAL SITUATION, I BELIEVE WE SHOULD CHOOSE THE SOLUTION THAT DECAYS AS $x \rightarrow \infty$. FOR $\omega > 0$, THIS ~~MEANS~~ IMPLIES $A=0$:

$$e^{-\sqrt{\frac{\omega}{2k}}(1+i)x} = e^{-\sqrt{\frac{|\omega|}{2k}}(1+i)x}$$

BUT WHEN $\omega < 0$, OUR SOLUTION BECOMES: (TAKING $\omega \rightarrow -|\omega|$)

$$\begin{aligned} X(x) &= A e^{i\sqrt{\frac{|\omega|}{2k}}(1+i)x} + B e^{-i\sqrt{\frac{|\omega|}{2k}}(1+i)x} \\ &= \underbrace{A e^{\sqrt{\frac{|\omega|}{2k}}(-1+i)x}}_{\text{DECAYING TERM}} + \underbrace{B e^{\sqrt{\frac{|\omega|}{2k}}(1-i)x}}_{\text{GROWING TERM}} \Rightarrow \underline{\underline{B=0}} \end{aligned}$$

PUTTING IT ALL TOGETHER,

$$\boxed{X(x) = e^{-\sqrt{\frac{|\omega|}{2k}}(1 + \text{sgn}(\omega) i)x}$$

THIS IS EQUIVALENT TO CHOOSING A SECOND BOUNDARY CONDITION,

$$\frac{\partial}{\partial x} u(x, t) \rightarrow 0 \text{ AS } x \rightarrow \infty.$$

Q: WHY DID I NOT SAY $u(x, t) \rightarrow 0$ AS $x \rightarrow \infty$?

Q: WHAT IS THE PHASE VELOCITY (ω/k)?

HOW DEEP SHOULD I BURY MY PIPES SO THAT THEY DON'T FREEZE IN THE WINTER?

HEAT DIFFUSION EQUATION:

$$\frac{\partial T}{\partial t} = \frac{\eta}{\rho C_p} \nabla^2 T = K \nabla^2 T$$

ASSUMING ANNUAL MEAN TEMPERATURE $> 0^\circ\text{C}$,

$$d \gg \sqrt{\frac{2K}{\omega}}$$

SOIL THERMAL PROPERTIES: (APPROXIMATE!)

THERMAL CONDUCTIVITY $\eta \sim 0.5 \text{ W/m}\cdot\text{K}$

DENSITY $\rho \sim 2000 \text{ kg/m}^3$ (2x H_2O)

HEAT CAPACITY $C_p \sim 1000 \text{ J/kg}\cdot\text{K}$

$$\Rightarrow K \sim 2.5 \times 10^{-7} \text{ m}^2/\text{s}$$

$$\omega = \frac{2\pi}{\text{YEAR}} = 2 \times 10^{-6} / \text{s}$$

$$\therefore d \gg \sqrt{\frac{2K}{\omega}} = 0.5 \text{ m.}$$